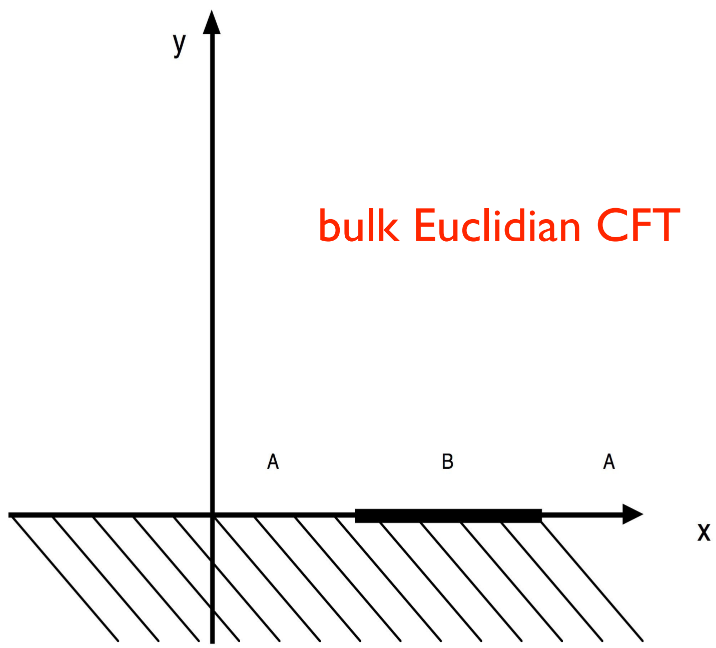
Exact overlaps in the anisotropic Kondo problem

by H. Saleur

Work with S. Lukyanov, J.L. Jacobsen and R. Vasseur [Phys. Rev. Lett. 114, 080601 (2015)]

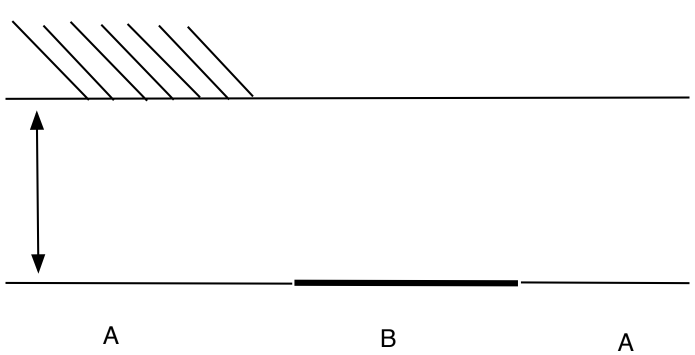
Anderson orthogonality again

- Modern point of view: Conformal boundary conditions changing operators



$$\langle O_{AB}(x_1) O_{BA}(x_2) \rangle = \frac{1}{|x_1 - x_2|^{2d_{AB}}}$$

Logarithmic mapping



$$w = \frac{iL}{\pi} \ln z$$

correlation function on the strip:

$$\langle O_{AB}(w_1, \bar{w}_1) O_{BA}(w_2, \bar{w}_2) \rangle \approx \left(\frac{\pi}{L}\right)^{2d_{AB}} \exp\left[-\frac{\pi}{L} d_{AB} |w_1 - w_2|\right]$$

length of the B interval

Calculate the same correlation function in a Hamiltonian formalism (in imaginary time)

$$|{}_{AA}\langle 0|0\rangle_{AB}| = \left(\frac{\pi}{L}\right)^{d_{AB}}$$

$$d_{AB} = \frac{L}{\pi}\left(E^0_{AB} - E^0_{AA}\right)$$

(Cardy, Affleck Ludwig)

Forgetting about the left boundary condition (fixed in what follows): we see that ground states with different conformal boundary conditions are orthogonal. This is the same as the Anderson Orthogonality catastrophe (Anderson 67) using

- single impurity in 3D gapless + isotropy and reduction to s waves
- giving equivalence to 1 dim quantum Hamiltonian
- and equivalence of impurity fixed point with conformal boundary condition in the 2 dim Euclidian

■Anderson Orthogonality catastrophe (Anderson 67)

has nothing to do with interactions. Can be understood simply for free fermions (Landau Fermi liquid) as a collective effect : cumulated phase shift of all the one electron states hidden in the Fermi sea.

⟨○ ○ ○ ○○ ○|○○○○○○○⟩ → $$|_{AA}\langle 0|0\rangle_{AB}| \approx \exp\left[-\frac{1}{2\pi^2}\int \frac{d\omega}{\omega}(\delta_A(\omega)-\delta_B(\omega))^2]\right]$$

energy above Fermi surface   phase shifts

or $$|_{AA}\langle 0|0\rangle_{AB}| \propto L^{-(\delta_A^F-\delta_B^F)^2/2\pi^2}$$

phase shifts right at Fermi surface

■The catastrophe in the (anisotropic) Kondo problem: phase shifts at the Fermi surface with and without the Kondo impurity differ by $\delta=\frac{\pi}{2}$. The ground states with and without the impurity are orthogonal:

The impurity coupling (the Kondo temperature) is therefore non perturbative .

It is useful to think more about energy scales. While in general anisotropic Kondo is not a one fermion problem, we can think of the particular (Toulouse) anisotropy where it is. There, the interaction on the boundary, means the fermions do not see conformal boundary conditions and have an energy dependent phase shift:

$$\delta = \frac{\pi}{2} \xrightarrow[\omega]{} \delta = 0$$

$$e^{2i\delta} = i \tanh\left(\frac{\theta - \theta_K}{2} - \frac{i\pi}{4}\right) \qquad \omega = \mu e^{\theta} \qquad T_K = \mu e^{\theta_K}$$

(Kondo temperature)

If one of the Kondo temperatures is zero (no Kondo coupling), shifts at the Fermi surface differ by $\frac{\pi}{2}$

$$\left|{}_{T_K^{(2)}}\langle vac|vac\rangle_{T_K^{(1)}=0}\right| \propto L^{-\frac{1}{8}}, \quad \text{free case}$$

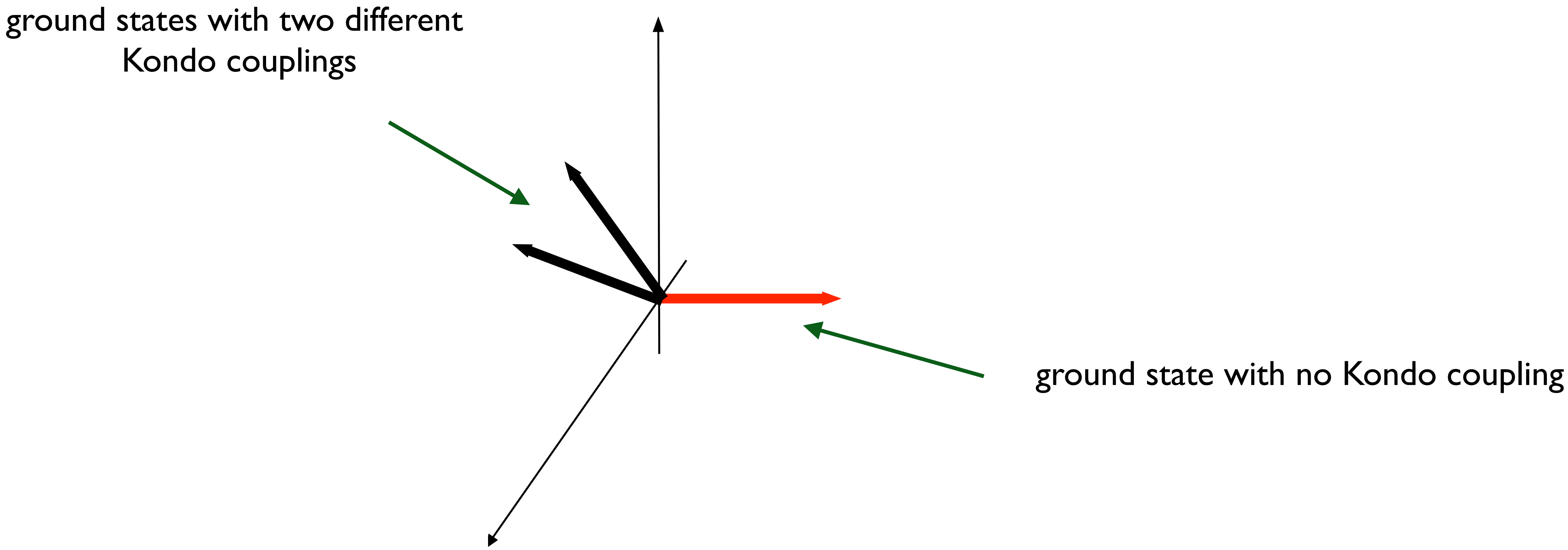
If both Kondo temperatures are non zero (two different, non zero values of the Kondo coupling), shifts at the Fermi surface are both equal to $\delta = \frac{\pi}{2}$ so there is a non zero overlap for the corresponding ground states

If one of the Kondo temperatures is much greater than the other:

orthogonality exponent

$$\left|{}_{T_K^{(2)}}\langle vac|vac\rangle_{T_K^{(1)}}\right| \propto \left(\frac{T_K^{(2)}}{T_K^{(1)}}\right)^{-\frac{1}{8}}, \quad T_K^{(2)} >> T_K^{(1)} \quad \text{free case}$$

- In general, and in the scaling limit such a scalar product is a universal function of the ratio $T_K^{(2)}/T_K^{(1)}$ (it is not perturbative in either of these temperatures, and it's not a one electron problem either)



what is this function in the free case, in the interacting case?

The set-up

- Anisotropic Kondo: 3D spinful Fermi liquid interacting with localized magnetic impurity. Spherical waves + reduction to s mode + bosonization + decoupling of the charge degrees of freedom + SU(2) interaction broken down to U(1) leaves

$$H = \frac{1}{2}\int_{-\infty}^{0}\left[(\partial_x\Phi)^2 + \Pi^2\right] + J_\perp\left(e^{i\sqrt{2\pi}\Phi(0)}\sigma^- + e^{-i\sqrt{2\pi}\Phi(0)}\sigma^+\right) + \frac{J_z}{2}\Pi(0)\sigma^z$$

Unfolding + canonical transformation

chiral boson

$$U_{J_z}^\dagger H U_{J_z} = \int dx(\partial_x\phi)^2 + J_\perp\left(e^{i\beta\phi(0)}\sigma_- + e^{-i\beta\phi(0)}\sigma_+\right)$$

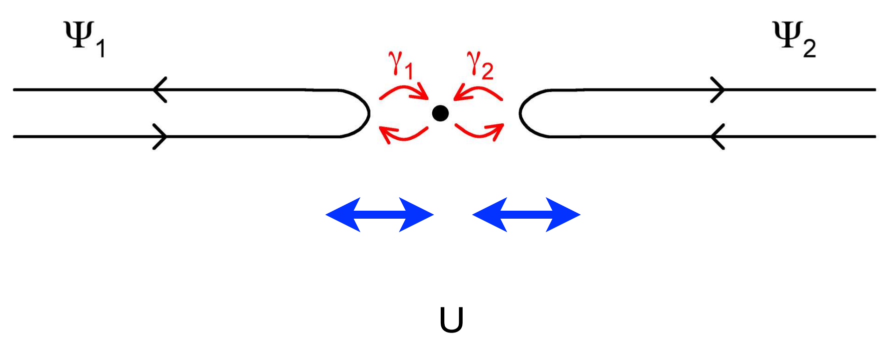
Dimension of the perturbation is

$$\frac{\beta^2}{8\pi} = \left(1 - \frac{J_z}{\sqrt{2\pi}}\right)^2$$

Crossover (Kondo) temperature: $T_K \propto J_\perp^{1/1-\frac{\beta^2}{8\pi}}$

what we want are the overlaps of the ground states of 1d Hamiltonians for two different values of the impurity coupling $J_\perp$

- This hamiltonian occurs in a variety of other contexts: two state problem in dissipative quantum mechanics, IRLM,...



- The scalar product is a particular case of matrix elements $\quad {}_{T_K^{(2)}}\langle \Psi^{(2)} | \Psi^{(1)} \rangle_{T_K^{(1)}}$

eigenstates

In the free (Toulouse) case: $\quad {}^{\epsilon_1,\ldots,\epsilon_n}_{T_K^{(2)}}\langle \theta_1, \ldots, \theta_n | \theta'_1, \ldots, \theta'_m \rangle^{\epsilon'_1,\ldots,\epsilon'_m}_{T_K^{(1)}}$

In general, one can use a similar description but the corresponding massless particles bear little relationship to the original electrons:

anisotropic Kondo is integrable

These matrix elements are crucial in the study of quenches: example of the Kondo exciton (absorption of a photon ~ turning on Kondo coupling)

In this context the overlap gives access to the probability for the system to remain in the ground state after a quench

The ratios $$\frac{{}_{T_K^{(2)}}^{\epsilon_1,\ldots,\epsilon_n}\langle\theta_1,\ldots,\theta_n|\theta'_1,\ldots,\theta'_m\rangle_{T_K^{(1)}}^{\epsilon'_1,\ldots,\epsilon'_m}}{{}_{T_K^{(2)}}\langle vac|vac\rangle_{T_K^{(1)}}}$$

can (in principle) be determined by an axiomatic form-factors approach

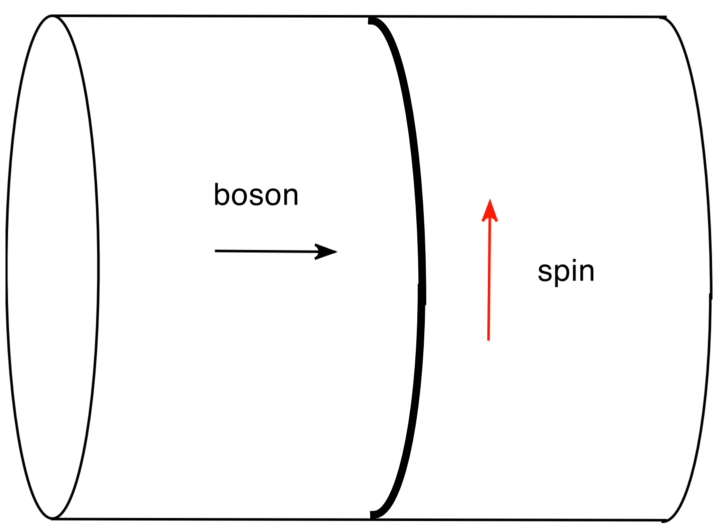
This was used to calculate the Loschmidt echo and the work distribution in the Kondo exciton problem (Vasseur, Trinh, Haas, Saleur 13)

Some information on $|{}_{T_K^{(2)}}\langle vac|vac\rangle_{T_K^{(1)}}|$ can then be obtained by resumming the series - not too efficient however.

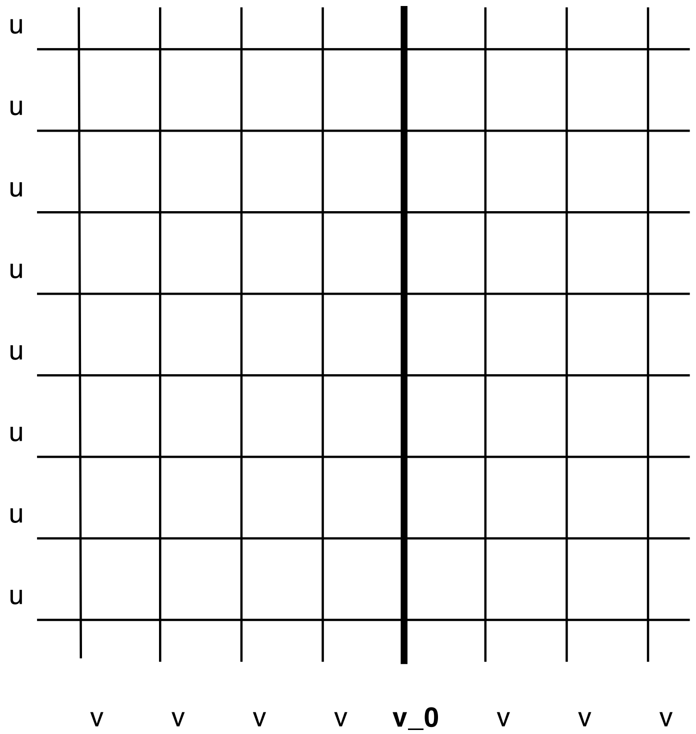
Can one get the overlaps directly and exactly? (Lesage Saleur 98)

Some ideas about the formalism

- In imaginary time, the insertion of the impurity can be thought of in terms of a monodromy matrix M. It acts on the spin degrees of freedom, and its elements are operators acting in the (right moving) free boson Hilbert space. (Bazhanov Lukyanov Zamolochikov 94)



- This is exactly the continuum limit of the six vertex model monodromy matrix, in the particular case of a vertical line carrying a large bare rapidity



The monodromy matrix can be expressed as

$$M(J_\perp) = e^{2i\pi P\sigma^z} \mathcal{P} \exp\left[J_\perp \int_0^{2\pi/T} \left(q^{\sigma^z/2} e^{i\beta\phi(0)} \sigma_- + q^{-\sigma^z/2} e^{-i\beta\phi(0)} \sigma_+ \right) \right] \qquad q = -e^{-8i\beta^2}$$

Integrability of Kondo arises in this context from the zero curvature representation of (classical) SG

$$(\partial_x^2 - \partial_t^2)\Phi + \frac{m^2}{\beta} \sin(\beta\Phi) = 0$$

$$[\partial_+ - A_+, \partial_- - A_-] = 0$$

with

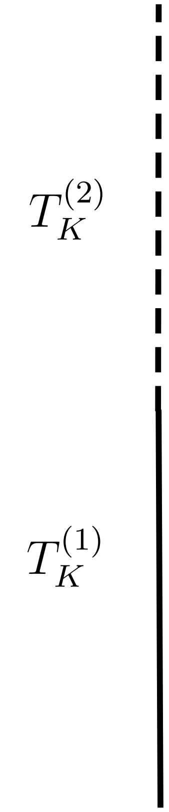
$$A_+ = \frac{1}{4i} \left[\beta(\partial_x\Phi + \partial_t\Phi)\sigma^3 - me^{\theta}(e^{-i\beta\Phi/2}\sigma^- + e^{i\beta\Phi/2}\sigma^+) \right]$$
$$A_- = \frac{1}{4i} \left[\beta(\partial_x\Phi - \partial_t\Phi)\sigma^3 + me^{-\theta}(e^{i\beta\Phi/2}\sigma^- + e^{-i\beta\Phi/2}\sigma^+) \right]$$

no anomalous dimension as this is classical

non chiral field

spectral parameter $me^{\theta} \propto J_\perp$

- The geometry we are interested in is



requires splitting the monodromy matrix into two objects propagating (in the vertical sense) from $y=-\infty$ to $y=0$ and from $y=0$ to $y=\infty$

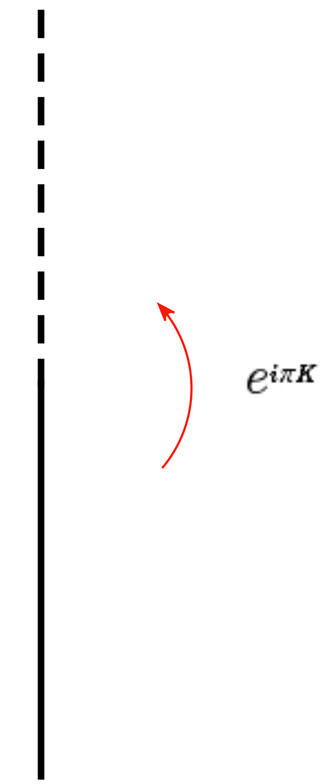
lattice vertex operators (Foda Jimbo Miwa)

In the classical case, one writes $M(\theta)=T(\theta)Q(\theta)$

Jost solutions

so we need a quantum version of the Jost functions (Lukyanov, Shatashvili 93,94)

- Note: in order to have T and Q act on the same space one needs to turn to radial quantization (corner transfer matrix)



- General arguments lead to $$T(\theta) = \begin{pmatrix} iT_+(\theta) & iT_-(\theta) \\ T_+(\theta+i\pi) & -T_-(\theta+i\pi) \end{pmatrix}$$

$$C^{ab} T_a(\theta + i\pi) T_b(\theta) = i, \quad C^{ab} = \delta_{a+b}$$

where now $me^{\theta} \propto J_{\perp}^{1/1-\frac{\beta^2}{2\pi}}$

now there is an anomalous dimension

we're interested in objects acting on the spin degrees of freedom, and which are in fact operators acting on the free boson Hilbert space

Relations satisfied by T

- Recall that anisotropic Kondo can be studied using massless scattering (massless limit of the soliton/antisoliton description of SG) (Faddeed Takhtajan, Andrei, Fendley, Fendley Saleur, Zamo^2...)

For R moving particles obeying $e = p = me^{\theta}$

$$Z_a^\dagger(\theta_1) Z_b^\dagger(\theta_2) = S_{ab}^{cd}(\theta_1 - \theta_2) Z_d^\dagger(\theta_2) Z_c^\dagger(\theta_1)$$

$$\begin{aligned} S_{++}^{++}(\theta) &= S(\theta) \\ S_{+-}^{+-}(\theta) &= S(\theta)\frac{\sinh\frac{\theta}{\xi}}{\sinh\frac{i\pi-\theta}{\xi}} \\ S_{+-}^{-+}(\theta) &= S(\theta)\frac{\sinh\frac{i\pi}{\xi}}{\sinh\frac{i\pi-\theta}{\xi}} \end{aligned}$$

$$\frac{\beta^2}{8\pi} \equiv \frac{\xi}{\xi+1}$$

$$S(\theta,\xi) = -\exp\left(-i\int_{-\infty}^{\infty}\frac{d\omega}{\omega}\sin\omega\theta\frac{\sinh[(\pi-\pi/\xi)\omega/2]}{2\sinh(\pi\omega/2\xi)\cosh(\pi\omega/2)}\right)$$

S matrix has quantum su(2) symmetry with $q_Z = e^{\frac{i\pi}{\xi}}$

- Massless kinks scatter on the Kondo impurity with (Andrei, Fendley)

Note: it does not depend on the (Kondo) anisotropy!

$$R(\theta) = i\tanh\left(\frac{\theta-\theta_0}{2} - \frac{i\pi}{4}\right) \qquad T_K = me^{\theta_0}$$

- (Conjectured) relations for ZF and Joost operators:

impurity scattering

$$\begin{aligned} Z_a^\dagger(\theta_1)T_b(\theta_2) &= ab\ i\ \tanh\left(\frac{\theta_1-\theta_2}{2} - \frac{i\pi}{4}\right) T_b(\theta_2)Z_a^\dagger(\theta_1) \\ T_a(\theta_1)T_b(\theta_2) &= R_{ab}^{cd}(\theta_1-\theta_2)T_d(\theta_2)T_c(\theta_1) \end{aligned}$$

The lattice scattering matrix

with

$$\begin{aligned} R_{++}^{++}(\theta) &= R(\theta) \\ R_{+-}^{+-}(\theta) &= -R(\theta)\frac{\sinh\frac{\theta}{\xi+1}}{\sinh\frac{i\pi-\theta}{\xi+1}} \\ R_{+-}^{-+}(\alpha) &= R(\theta)\frac{\sinh\frac{i\pi}{\xi+1}}{\sinh\frac{i\pi-\theta}{\xi+1}} \end{aligned}$$

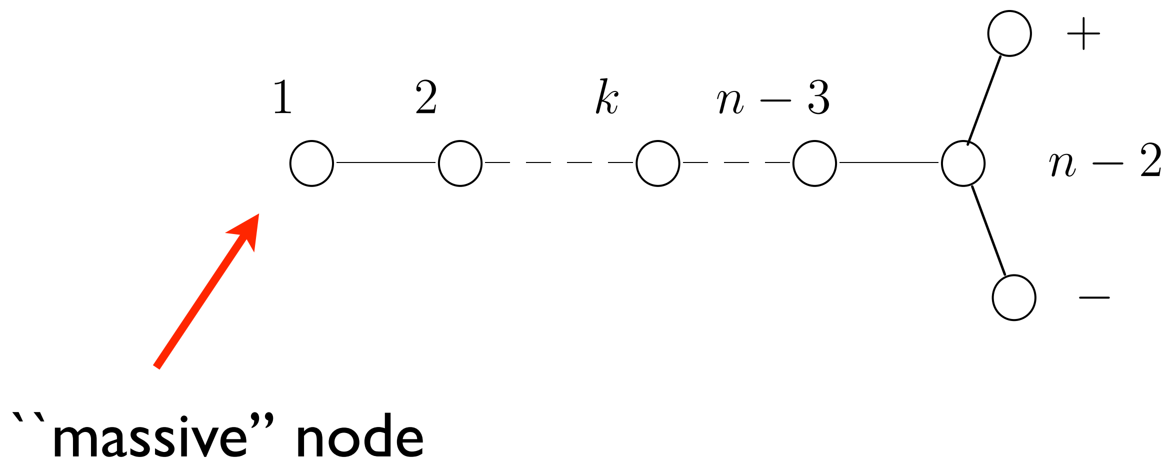
quantum su(2) symmetry with $q_M = e^{\frac{i\pi}{\xi+1}} = q$

$$R(\theta) = -S(-\theta, \xi+1)$$

we treat impurities just like one more type of particle; the Kondo coupling is traded for a ‘rapidity’

- The ZF and Joost operators have different quantum group symmetries $q_Z = e^{\frac{i\pi}{\xi}}$ $q_M = e^{\frac{i\pi}{\xi+1}} = q$

like the SG S matrices and the 6 vertex R matrices. Can be seen from the TBA ($\xi = n-1$)



The main result

- Bosonization of form factors (Lukyanov) leads to ($T_K^{(2)}/T_K^{(1)} = e^{\theta_{21}}$)

$$\left|{}_{T_K^{(2)}}\langle vac|vac\rangle_{T_K^{(1)}}\right| = \langle T_\pm(\theta_1)T_\mp(\theta_2+i\pi)\rangle = (1+\xi)\,\frac{\sinh\frac{\theta_{12}}{2(1+\xi)}}{\sinh\frac{\theta_{12}}{2}}\;G(\theta_{12})$$

with minimal solution

$$G(\theta) = \exp\left[\int_0^\infty \frac{dt}{t}\frac{\sin^2(\theta t/\pi)}{\sinh 2t\cosh t}\frac{\sinh t\xi}{\sinh t(\xi+1)}\right]$$

"minimal" soliton soliton form factor, but for a renormalized SG coupling!

symmetry $\theta_{12} \to \theta_{21}$

- Checks

$$\left|{}_{T_K^{(2)}}\langle vac|vac\rangle_{T_K^{(1)}}\right| \propto \exp\left[-\frac{\xi}{4(\xi+1)}\theta_{21}\right], \quad \theta_{21}\to\infty$$

so

$$\left|{}_{T_K^{(2)}}\langle vac|vac\rangle_{T_K^{(1)}}\right| \propto \left(\frac{T_K^{(2)}}{T_K^{(1)}}\right)^{-\frac{\beta^2}{32\pi}}, \quad T_K^{(2)} >> T_K^{(1)}$$

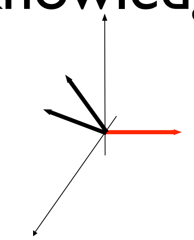
and

$$h_{Kondo} = \frac{1}{4}\frac{\beta^2}{8\pi}$$

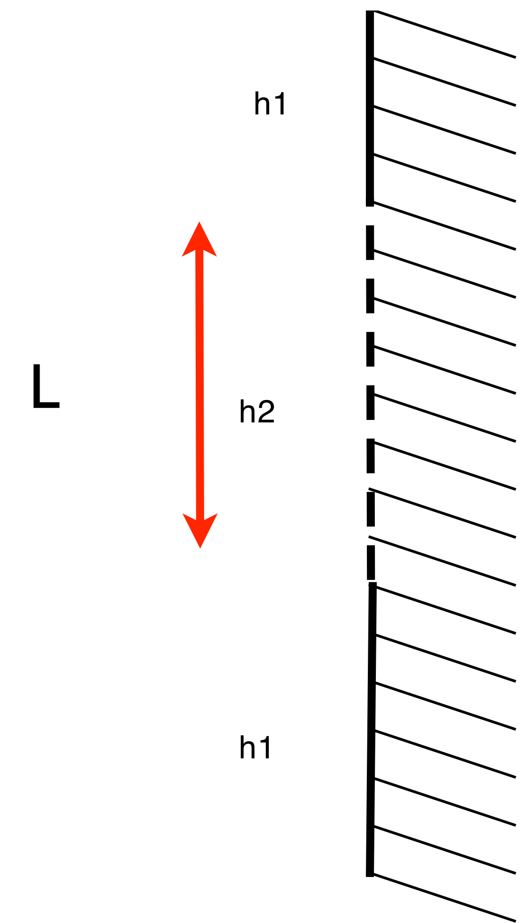
dimension of the boundary condition changing operator from weak to strong coupling Kondo fixed point

- Perturbative calculations require both Kondo couplings to be non zero to avoid the catastrophe.

The expansion variable is then $\frac{J_\perp^{(2)}-J_\perp^{(1)}}{J_\perp^{(1)}}$ requiring knowledge of correlation functions for non zero Kondo coupling to start with!



Can be done in the free fermion case where the calculation can be reformulated in terms of an Ising model with two different boundary fields. The scalar product is essentially the term of order one in L for the partition function



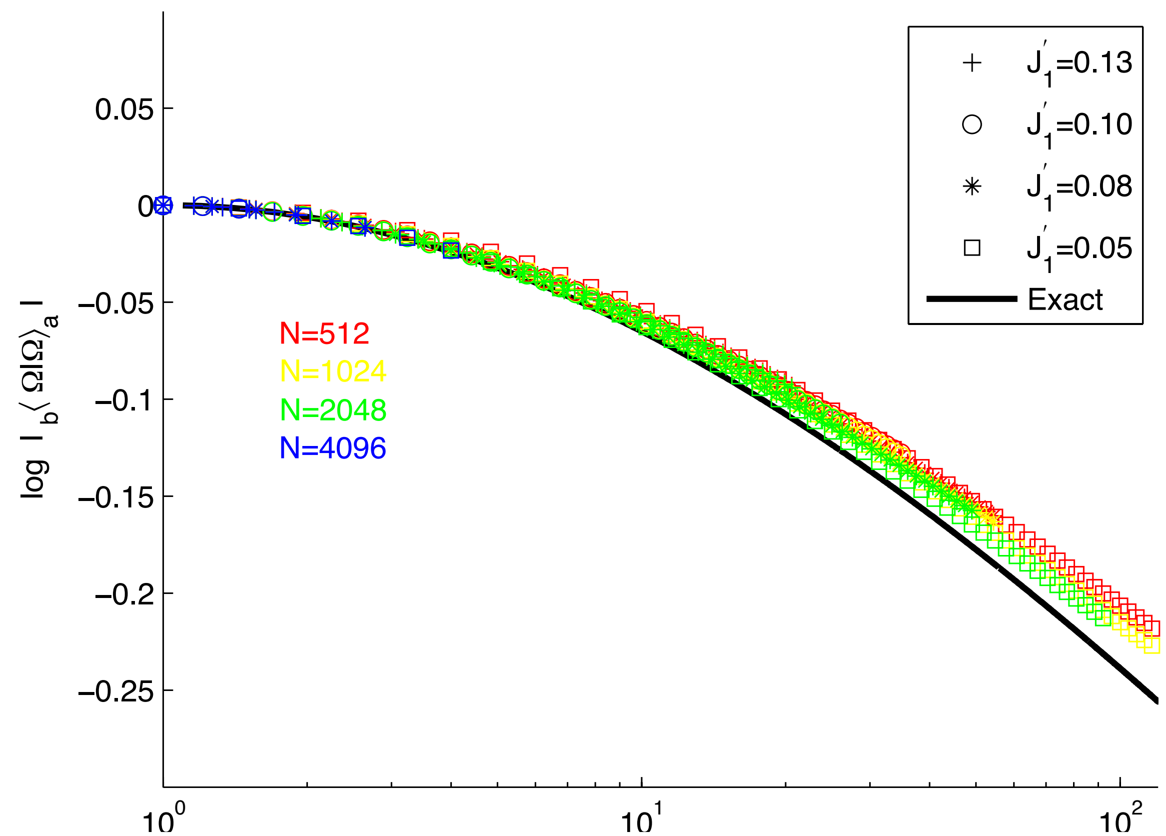
$$\left|{}_{T_K^{(2)}}\langle vac|vac\rangle_{T_K^{(1)}}\right| = 1-\frac{\theta_{21}^2}{8\pi^2}, \quad \theta_{12} << 1$$

Can also be done in the semiclassical case

$$\left|{}_{T_K^{(2)}}\langle vac|vac\rangle_{T_K^{(1)}}\right| = 1+\frac{\xi}{2}-\frac{\xi}{4}\theta_{12}\coth\frac{\theta_{12}}{2}+(\xi^2) \qquad \xi \approx \frac{\beta^2}{8\pi}$$

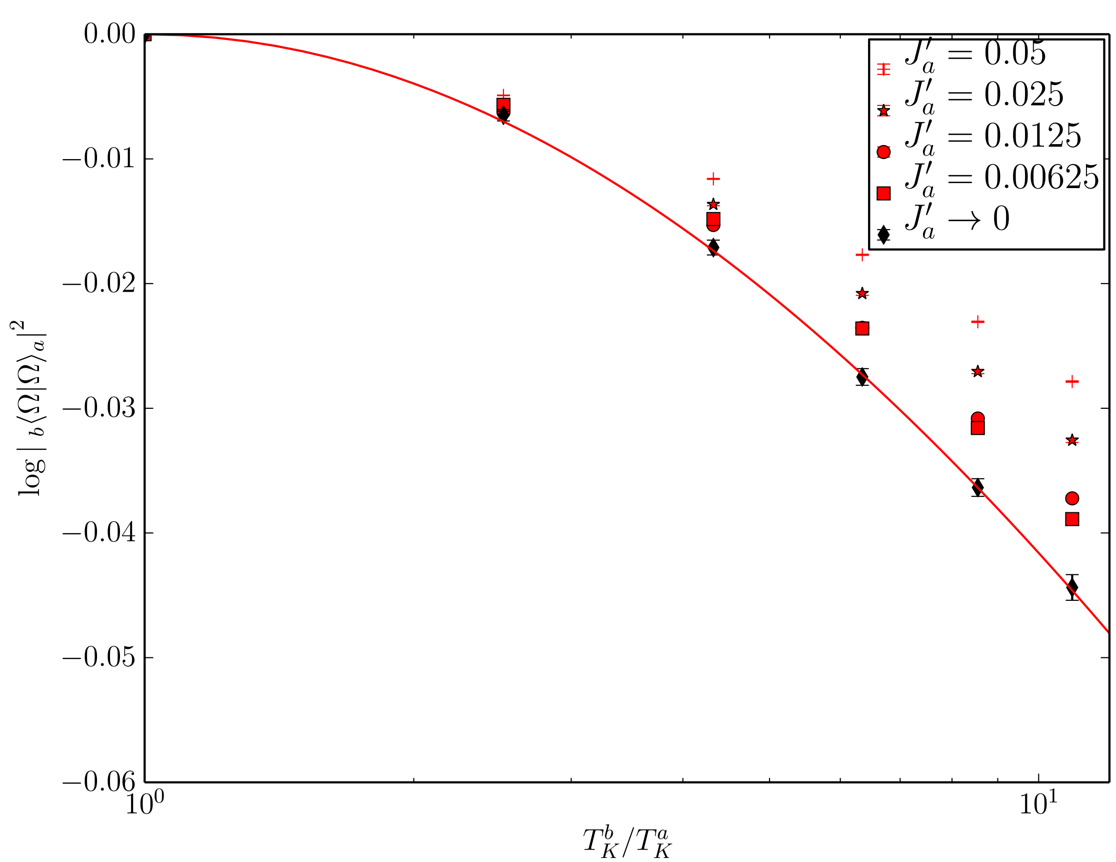
Numerics: difficult because scalar product evolves slowly, and finite size effects are very big (bare coupling must be very small, but Kondo length much smaller than system size!)

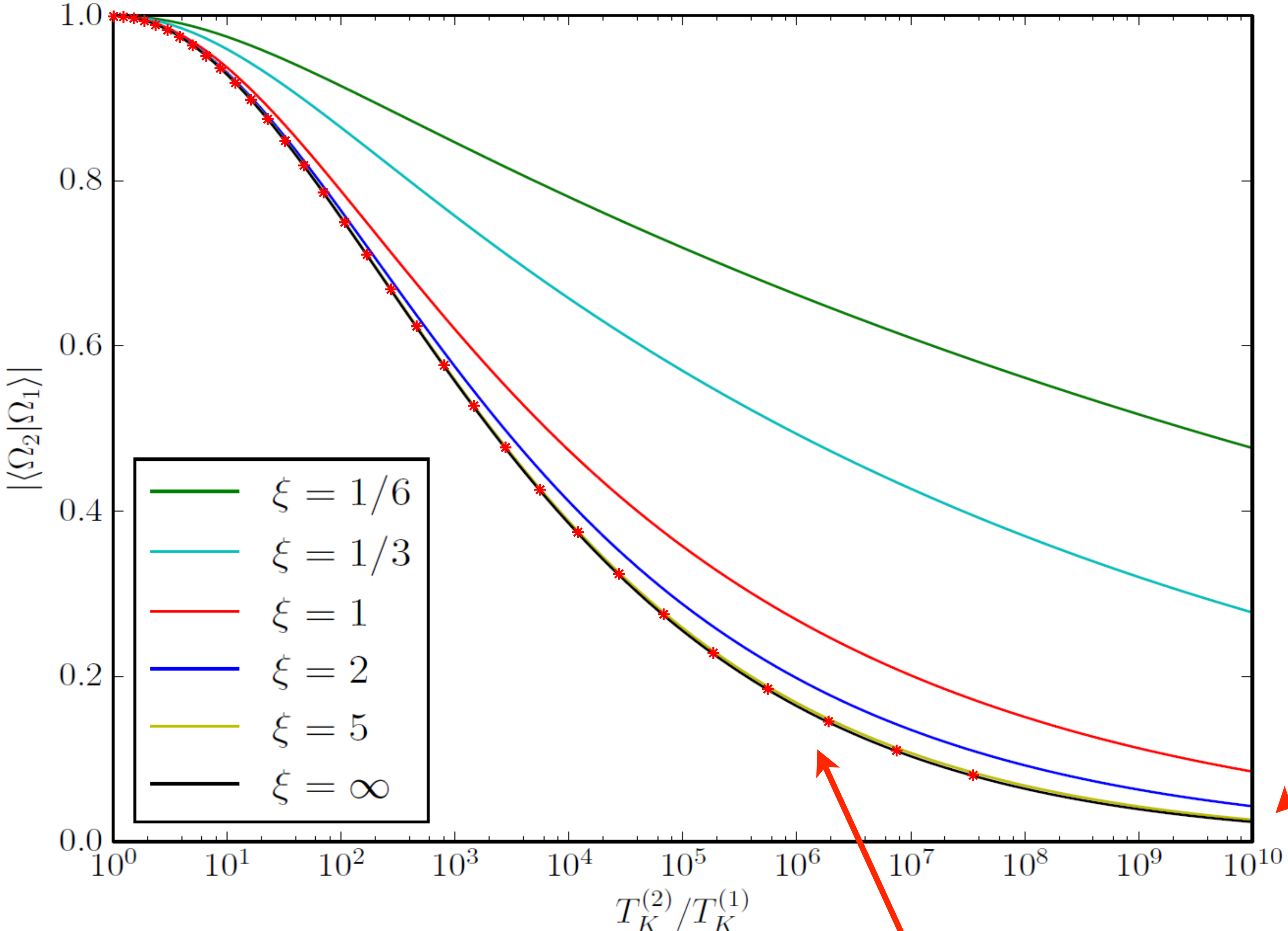
free case:



$h = 1/4$

Double extrapolation required.
Length up to 800





scale!

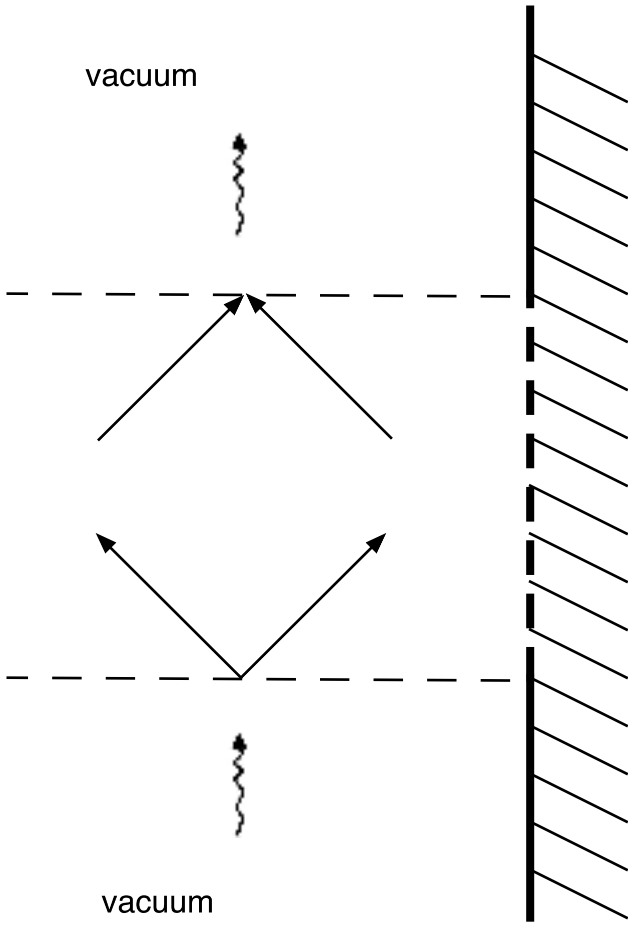
NRG data courtesy of A. Weichselbaum

Other overlaps: Form factors for ‘BC changing operators’

- Follow by ‘ordinary axiomatic approach’

Eg leading diagram for Loschmidt echo:

$$\frac{{}_{T_K^{(2)}}^{\epsilon_1,\ldots,\epsilon_n}\langle \theta_1,\ldots,\theta_n | \theta_1',\ldots,\theta_m' \rangle_{T_K^{(1)}}^{\epsilon_1',\ldots,\epsilon_m'}}{{}_{T_K^{(2)}}\langle vac | vac \rangle_{T_K^{(1)}}}$$



leading to work distribution etc.

Note that the ratios are well defined in the conformal limit, even if scalar products all vanish. Example at the Toulouse point (Ising model)

$$\frac{{}_+\langle\theta_1,\theta_2|vac\rangle_f}{{}_+\langle vac|vac\rangle_f} = i\,\tanh\,\frac{\theta_{12}}{2} \quad = \frac{0}{0}$$

free/fixed BC

so a naive check of unitarity (say) leads to

$${}_f\langle vac|vac\rangle_f = |{}_+\langle vac|vac\rangle_f|^2 \left(1 + \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\tanh^2\frac{\theta_{12}}{2} + \dots\right)$$

0 infinity

In general the approach remains plagued by IR divergences: Anderson catastrophe strikes back!

For instance the Loschmidt echo (in imaginary time) for a quench at the Toulouse point will involve

$$ {}_f\langle vac|e^{-H_{T_b}\tau}|vac\rangle_f = \sum_{n=0}^{\infty}\int \frac{d\theta_i}{2\pi} e^{-\tau m \sum e^{\theta_i}} |_{T_b}\langle \theta_1,\ldots,\theta_n|vac\rangle_f|^2 $$

can be calculated by writing it as

$$ \frac{\sum_{n=0}^{\infty}\int \frac{d\theta_i}{2\pi} e^{-\tau m \sum e^{\theta_i}} \frac{|_{T_b}\langle \theta_1,\ldots,\theta_n|vac\rangle_f|^2}{|_{T_b}\langle vac|vac\rangle_f|^2}}{\text{same with } \tau=0} $$

known from FF axioms

IR divergences can be subtracted
by simultaneous expansion of numerator
and denominator

leading to (Vasseur et al. 2013)

$$\int_0^\infty \frac{\mathrm{d}u}{2\pi u}(\mathrm{e}^{-xu}-1)\Psi(u)+\frac{1}{2!}\int_0^\infty \frac{\mathrm{d}u_1}{2\pi u_1}\int_0^\infty \frac{\mathrm{d}u_2}{2\pi u_2}(\mathrm{e}^{-x(u_1+u_2)}-1)\left(\left(\frac{u_1-u_2}{u_1+u_2}\right)^2-1\right)\Psi(u_1)\Psi(u_2)$$

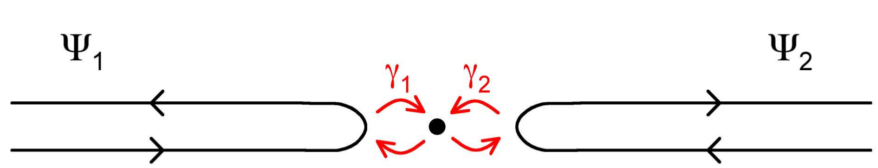
$$+\frac{1}{3!}\int_0^\infty \frac{\mathrm{d}u_1}{2\pi u_1}\int_0^\infty \frac{\mathrm{d}u_2}{2\pi u_2}\int_0^\infty \frac{\mathrm{d}u_3}{2\pi u_3}(\mathrm{e}^{-x(u_1+u_2+u_3)}-1)\left[\left(\frac{u_1-u_2}{u_1+u_2}\right)^2\left(\frac{u_1-u_3}{u_1+u_3}\right)^2\left(\frac{u_2-u_3}{u_2+u_3}\right)^2+2\right.$$
$$\left.-\left(\frac{u_1-u_2}{u_1+u_2}\right)^2-\left(\frac{u_1-u_3}{u_1+u_3}\right)^2-\left(\frac{u_2-u_3}{u_2+u_3}\right)^2\right]\Psi(u_1)\Psi(u_2)\Psi(u_3)+\ldots$$

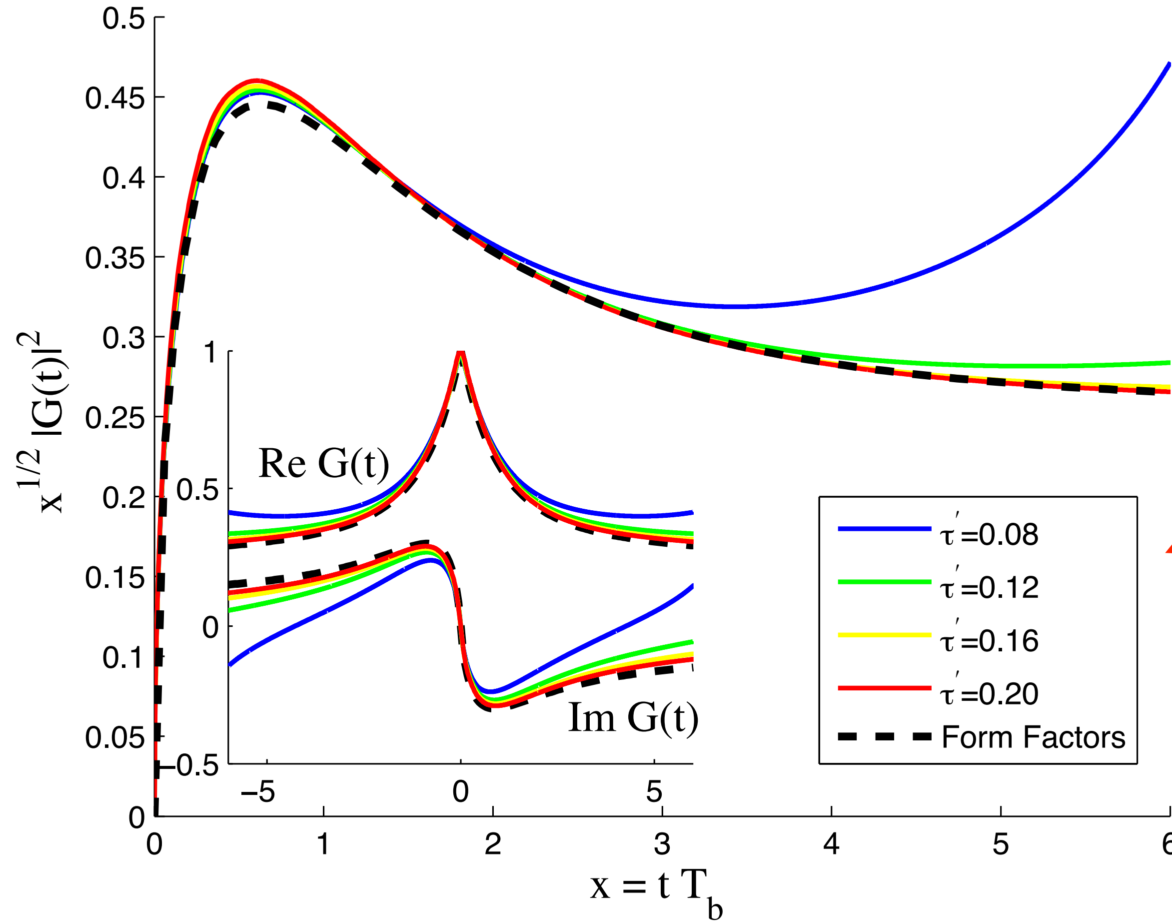
$$x=\tau T_b$$

where

$$\Psi(u)=\frac{\sqrt{u}}{1+u^2}\exp\left[\int_{-\infty}^\infty \frac{\mathrm{d}t}{2t}\left(\frac{2}{t}-\frac{\cos\frac{\ln u}{2\pi}t}{\cosh\frac{t}{4}\sinh\frac{t}{2}}\right)\right],$$

- This converges also in real time, giving access eg to the Loschmidt echo for a sudden quench in the RLM

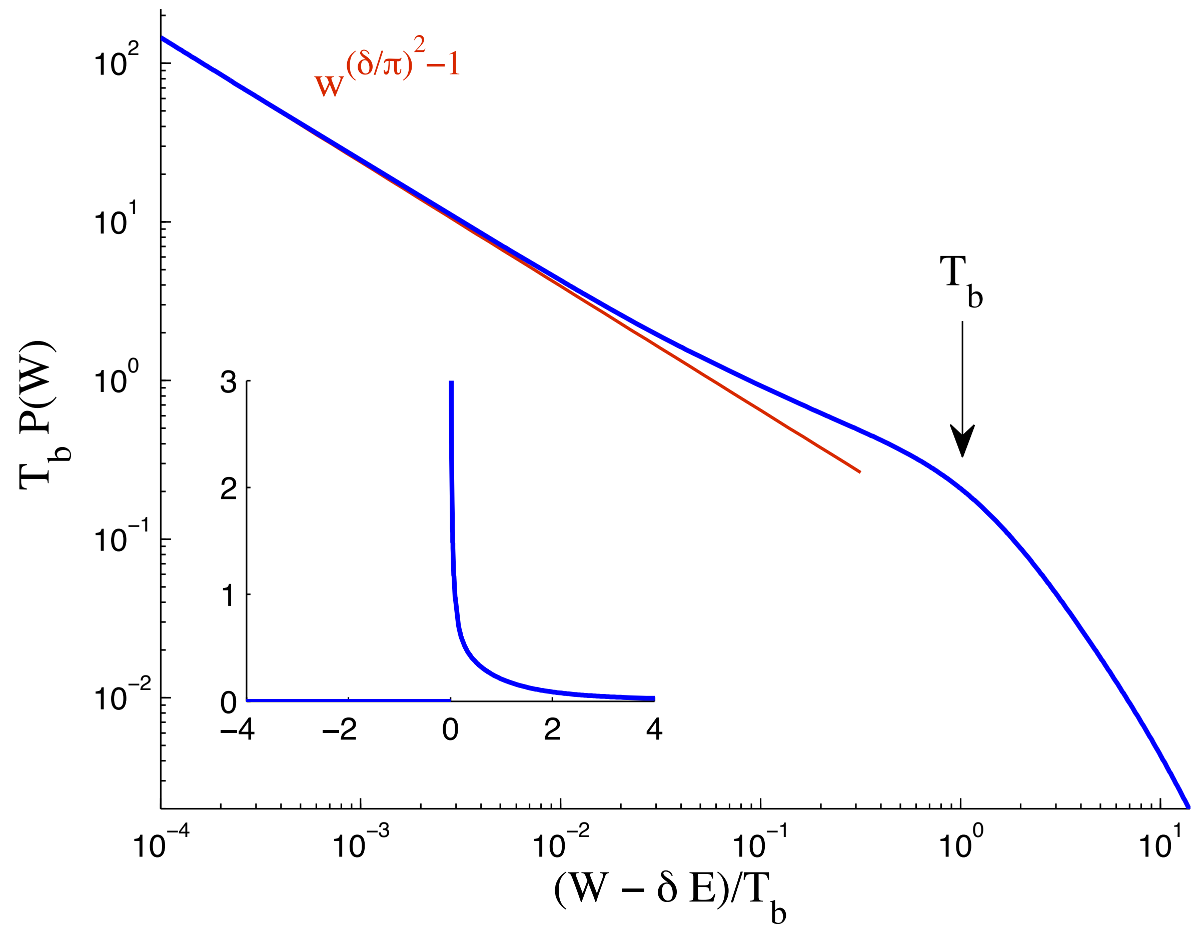




quenched tunneling amplitude, which is the same as the bare Kondo coupling

$|_f\langle vac|e^{-iH_{T_b}t}|vac\rangle_f| \propto t^{-1/4}$ at large times follows from CFT (orthogonality exponent again)

the work distribution then has a bump around the Kondo temperature (Tureci et al. 2011)



Power law singularity at small W

$$P(W) \underset{W \ll T_b}{\propto} \frac{1}{T_b}\theta(W)\left(\frac{T_b}{W}\right)^{3/4}$$

Conclusion

- While a lot of things are possible, things are not particularly pretty in general.
- This kind of approach starts from excitations over the physical vacuum, ie a vacuum filled with a large number of particles. Alternative approaches start from bare vacuum - better suited to different problems (N. Andrei)
- The overlap of ground states is an intriguing exception. Hints at more structure (differential equations), bypassing the Bethe ansatz.