Loop models and Liouville at c<1

by H. Saleur

Work in progress with Y. Ikhlef and J.L. Jacobsen,
based on old discussions with Alyosha Z.

Motivations and apologies

- There's a bunch of physics problems which can be reformulated in terms of 'non hermitian' quantum mechanics in 1+1 dimensions. In gapless, relativistic invariant cases, the long distance limit gives rise to LCFTs.

- A vast class of examples corresponds to spin chains on supergroups, or q-deformations thereof.
- In many cases, non-unitarity can be traced back to a mild non-locality. But the latter is not essential. Superspin chains are perfectly local.

- The idea of the algebraic associative approach [Read Saleur, Pearce Rasmussen Zuber] is to tackle the difficulty of LCFTs using finite dimensional lattices. It is perfectly reasonable physically since the only good way to define strongly interacting field theories is via lattice regularizations anyhow.

- The algebraic approach encounters terrible obstacles. The bulk case is still in its infancy [Gainutdinov Read Saleur Vasseur 15]. All the nice fusion calculations have provided… no concrete result whatsoever (in terms of calculating something useful, that is measurable)

so we shall dirty our hands!!!

and this talk will be very non mathematical

The ordinary Liouville CFT

- One of the simplest examples of non rational, unitary CFT [Zamolodchikov^2, Teschner, Dorn Otto]

$$\mathcal{L} = \frac{1}{4\pi}(\partial_a \phi)^2 + \mu e^{2b\phi}$$

+ curvature terms, e.g. on Riemann sphere $\phi(z,\bar{z}) \sim -Q \ln(z\bar{z}), \quad |z| \to \infty$

exactly marginal

$$T = -(\partial\phi)^2 + Q\partial^2\phi$$ with $c = 1 + 6Q^2$, $Q = b + \frac{1}{b}$

- Region most studied is $c \geq 25$ so b is real

Vertex operators $V_\alpha = e^{2\alpha\phi(z,\bar{z})}$ with $\Delta = \alpha(Q-\alpha)$

Spectrum is continuous $\alpha = \frac{Q}{2} + iP, \quad \Delta = \frac{Q^2}{4} + P^2$

identity is not normalizable, corresponding to Z=infinity

The three point function (the DOZZ formula)

$$C(\alpha_1, \alpha_2, \alpha_3) = A_b \frac{\Upsilon_0 \Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_{123} - Q)\Upsilon(\alpha_{12}^3)\Upsilon(\alpha_{23}^1)\Upsilon(\alpha_{13}^2)}$$

(amplitude multiplying the usual factor determined by conformal invariance)

$\alpha_{12}^3 \equiv \alpha_1 + \alpha_2 - \alpha_3$ etc

- This implies a choice of normalization
- This involves the function $\Upsilon \equiv \Upsilon_b$

$$\ln \Upsilon_b(x) = \int_0^\infty \frac{dt}{t} \left[\left(\frac{Q}{2} - x \right)^2 e^{-t} - \frac{\sinh^2 \left(\frac{Q}{2} - x \right) \frac{t}{2}}{\sinh \frac{bt}{2} \sinh \frac{t}{2b}} \right]$$

- The formula works outside the spectrum

$$C(\alpha_1, \alpha_2, \alpha_3 \to 0) \sim 2\pi\delta(\alpha_1 + \alpha_2 - Q) + R(\alpha)\delta(\alpha_1 - \alpha_2)$$

with R the reflection amplitude $C(Q - \alpha_1, \alpha_2, \alpha_3) = R(\alpha_1) C(\alpha_1, \alpha_2, \alpha_3)$

A_b, R are known expressions

- The three point function is remarkable because it exists outside the naive domain

where Dotsenko Fateev screening is possible $\sum_i \alpha_i = Q - nb \quad \text{or} \quad \sum_i \alpha_i = Q - nb - mb^{-1}$

In fact the DF correlators occur as poles of the function C when the resonance conditions are met.

- So far 'ordinary Liouville' hasn't found statistical mechanics applications (except maybe [Kogan Mudry Tsvelik 95]). In particular, no spin chain is known whose continuum limit would be ordinary Liouville.

Analytic continuations of Liouville

- There's a variety of reasons to explore regions with b an arbitrary complex number [Harlow, Maltz, Witten 2011]

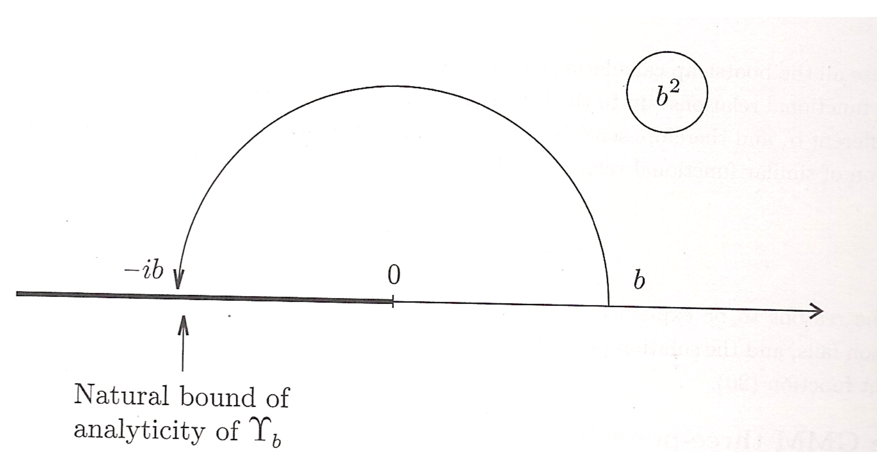
- Of particular interest is the region b purely imaginary $b = -i\hat{b}, \quad Q = i(\hat{b}^{-1} - \hat{b}) \equiv i\hat{Q}$

$$\mathcal{L} = \frac{1}{4\pi}(\partial_a \phi)^2 + \mu e^{-2i\hat{b}\phi} \qquad \text{and} \qquad c = 1 - 6\hat{Q}^2$$

$$V_{\hat{\alpha}} = e^{2i\hat{\alpha}\phi}, \quad \hat{\alpha} = \frac{\hat{Q}}{2} + p \qquad \Delta = \hat{\alpha}(\hat{\alpha} - \hat{Q}) = p^2 - \frac{\hat{Q}^2}{4}, \quad p \text{ real}$$

Note: the case p purely imaginary is usually referred to as time-like Liouville.

- The three point coupling admits analytic continuation in the complex b-plane but not to b purely imaginary



There is ‘another formula’ for b purely imaginary [Kostov Petkova, Zamolochikov, Schomerus]

$$\hat{C} = A_{\hat{b}} \frac{\Upsilon(\hat{b}+\hat{\alpha}_{13}^{2})\Upsilon(\hat{b}+\hat{\alpha}_{23}^{1})\Upsilon(\hat{b}+\hat{\alpha}_{12}^{3})\Upsilon(\hat{b}-\hat{Q}+\hat{\alpha}_{123})}{\left[\prod_{i=1}^{3}\Upsilon(\hat{b}+2\hat{\alpha}_i)\Upsilon(\hat{b}-\hat{Q}+2\hat{\alpha}_i)\right]^{1/2}}$$

where $\Upsilon \equiv \Upsilon_{\hat{b}}$ and $\hat{C}(0,\hat{\alpha},\hat{\alpha}) = 1$

This formula has unpleasant features:

$$\hat{C}(0,\hat{\alpha}_1,\hat{\alpha}_2) \neq 0 \quad \text{when} \quad \hat{\alpha}_1 \neq \hat{\alpha}_2!$$

$$\hat{C}(0,0,\hat{\alpha}) \neq 0!$$

Note: $\hat{C}(\hat{Q}-\hat{\alpha}_1,\hat{\alpha}_2,\hat{\alpha}_3) = \hat{C}(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\alpha}_3)$ while $\hat{C}$ is a totally symmetric function

For values corresponding to minimal models $c = 1 - 6\frac{(p-q)^2}{pq}$

and for the Dotsenko Fateev charges $\alpha = \alpha_{mn} = \frac{1-m}{2}\alpha_+ + \frac{1-n}{2}\alpha_-$

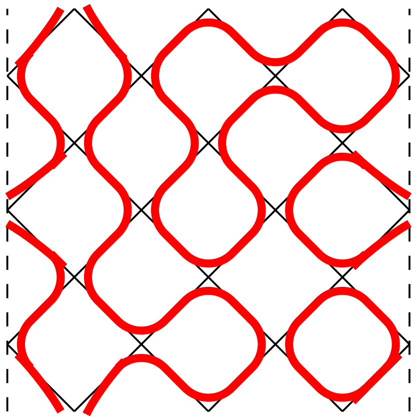
$\left(\alpha_+ = \hat{b}^{-1} > 1, \quad \alpha_- = -\hat{b}\right)$ it reproduces the minimal models three point couplings

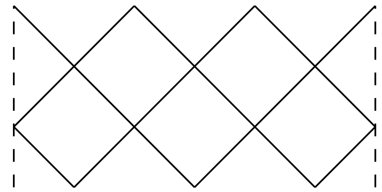
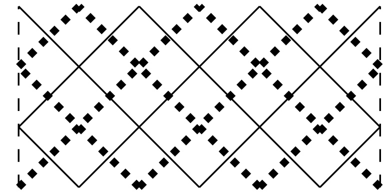
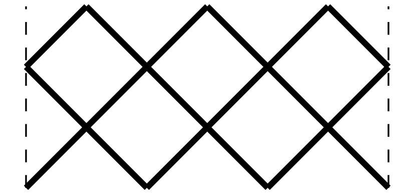
but not the fusion rules! (these require additional discrete factors)

What’s the meaning of all this?

Critical Loop models

- General set-up: draw self avoiding, mutually avoiding loops on a regular lattice. Fugacity β per unit length (monomer/edge) and fugacity n per loop.
- Interesting region is $n \in [-2, 2]$. There’s a critical point for $\beta = \beta_c$ and a critical dense phase for $\beta > \beta_c$. Both correspond to CFTs with $c < 1$
- A realization of the dense phase familiar to this audience is provided by the Temperley-Lieb loop model on the square lattice



$T_D =$  $\equiv$  $\equiv$ 

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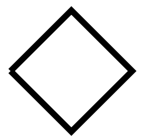
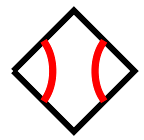
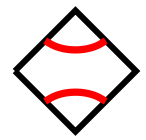
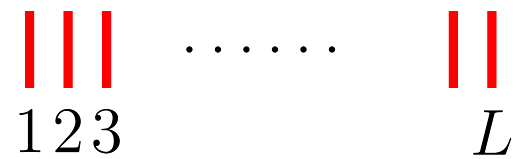
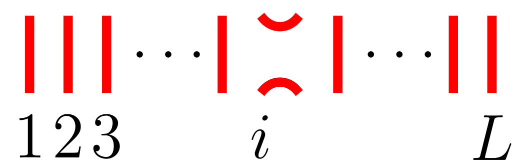
 $= x_1$  $+ x_2$ 

$$\check{R}_{i,i+1} = x_1 I + x_2 e_{i,i+1}$$

and our favorite TL algebra

$I =$ 

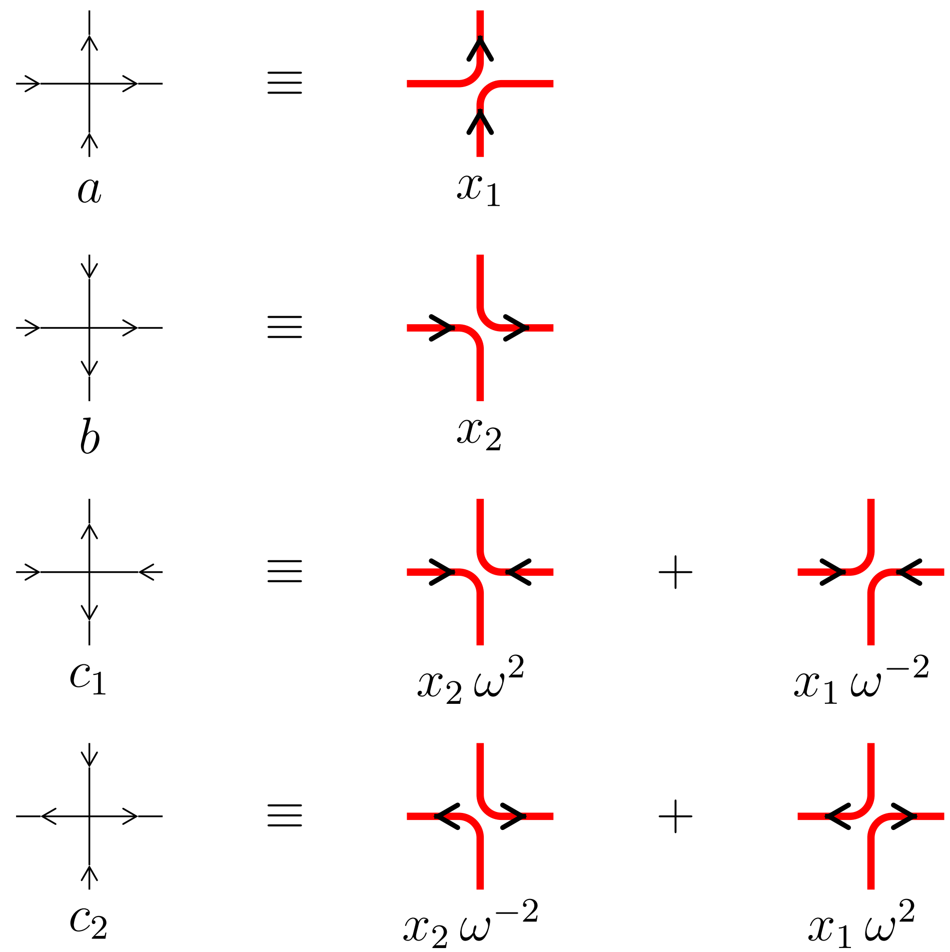
$e_{i,i+1} =$ 

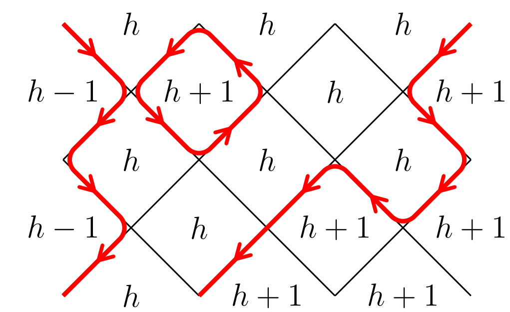
$e^2 = n\ e$ etc

the loop weight

We won't comment further about algebraic aspects here. But recall that loop models are a convenient way to tackle, for instance, super group invariant spin chains.

Non locality can be traded for non-unitarity by a map onto the 6 vertex model [Baxter]





Arrows are then interpreted as domain walls in a Solid on Solid model, whose long distance dynamics is described by a free boson. The weight loop n comes from summing over two orientations and giving a complex weight per left/right turn $n = \omega^4 + \omega^{-4}$

This induces as well a coupling to curvature, that is a charge at infinity [Den Nijs, Nienhuis].

Writing $n = -2\cos\pi g,\ g \in [0,1]$ one has

$$c = 1 - 6\frac{(1-g)^2}{g} \qquad \text{or} \qquad \hat{b} = \sqrt{g}$$

Of course conformal weights are obtained by a construction that matches Liouville. For instance consider the two point function defined by giving a weight $n_1 = 2\cos\pi e_1$ to loops separating a pair of points. This can be mapped onto the SOS model by introducing vertex operators and one easily finds

$$\Delta = \frac{e^2 - (1-g)^2}{4g} \equiv \hat{\alpha}(\hat{\alpha} - \hat{Q})$$

$$\hat{\alpha} = \frac{e}{2\sqrt{g}} + \frac{\hat{Q}}{2} = \frac{e}{2\sqrt{g}} + \frac{\hat{b}^{-1} - \hat{b}}{2}$$

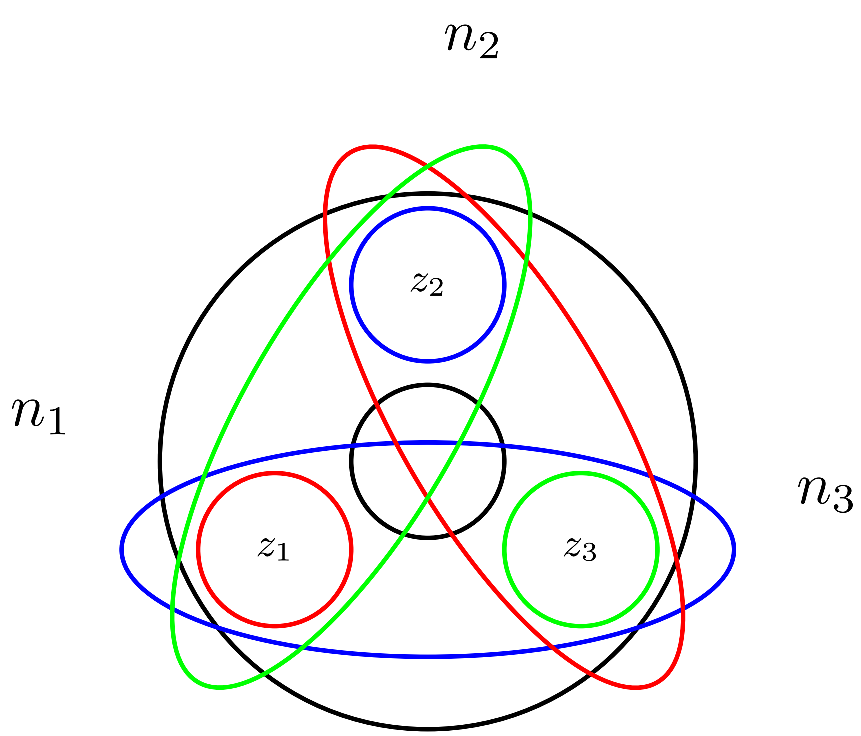
Note the symmetry $\hat{\alpha} \to \hat{Q} - \hat{\alpha}$ corresponds to $e \to -e.$

But is there more? Evidence: a lot of correlators can be defined that cannot be computed using the Dotsenko Fateev Coulomb gas. Can they be obtained using Liouville?

Note: in the CG & SOS mapping, the mere definition of the boson requires the height to be fixed at infinity. No zero mode integration! Invariance of the model by global shifts of height!

Three point couplings in loop models

- Our main claim. Consider a loop model with modified weight n_i for the loops separating point i from the other two (loops encircling none or all three points get weight n)



$$\hat{C}(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3) = Z_{123}\sqrt{Z_{000}\frac{Z_{011}}{Z_{101}Z_{110}}\frac{Z_{202}}{Z_{220}Z_{022}}\frac{Z_{330}}{Z_{033}Z_{303}}}$$

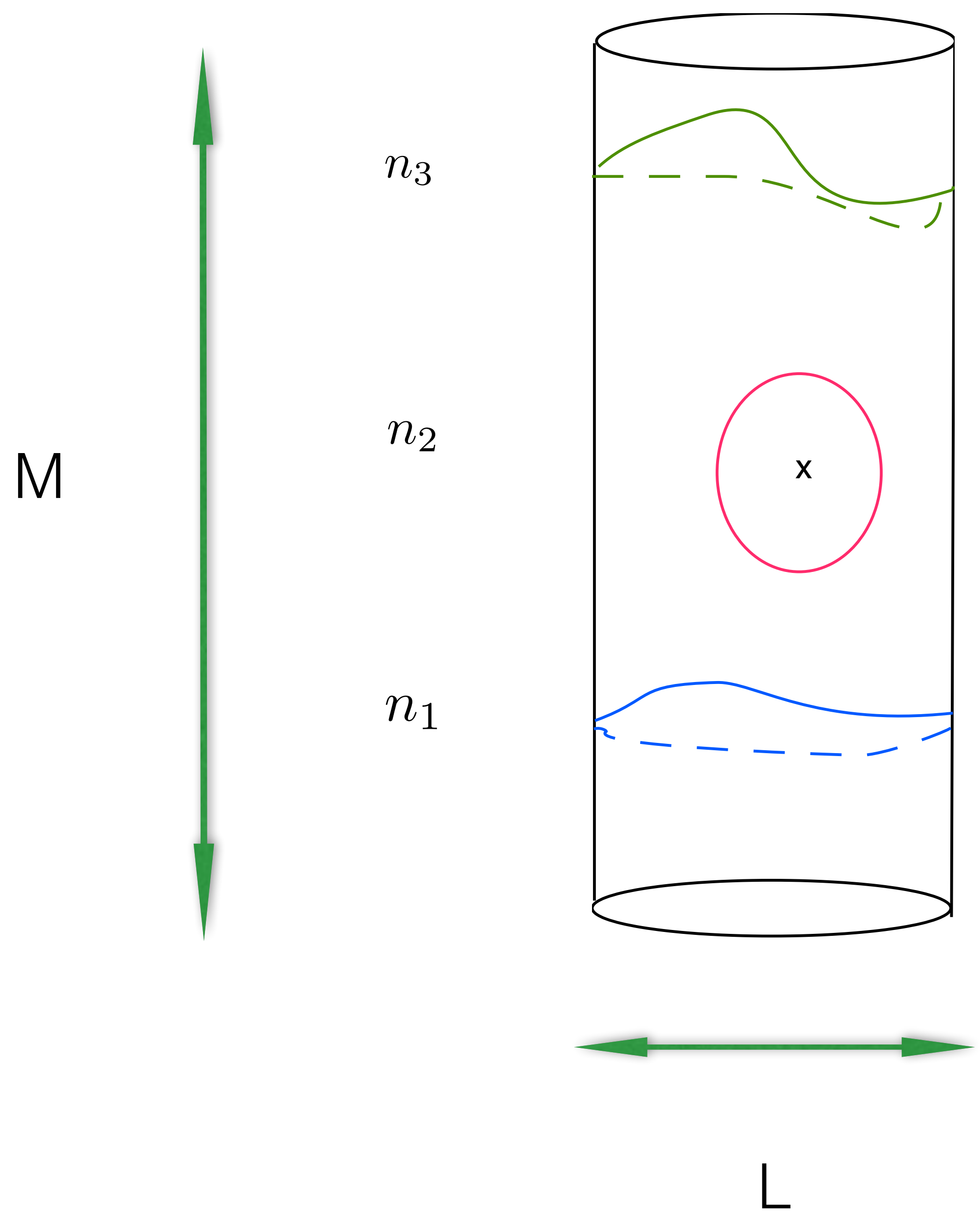
$$Z_{123} \equiv Z_{n_1 n_2 n_3}(1,2,3)$$

$$Z_{022} \equiv Z_{n,n_2,n_2}(1,2,3)$$

etc

- This result is totally unreachable by ‘Coulomb gas’ techniques, where only neutral combinations (including maybe the charge at infinity) make sense.

Now how to check this claim? Difficult in the plane, easier on the cylinder using transfer matrix calculations



$$\hat{C}(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3) = \frac{Z_{123}}{Z_{220}} \sqrt{\frac{Z_{202} Z_{000}}{Z_{101} Z_{303}}}$$

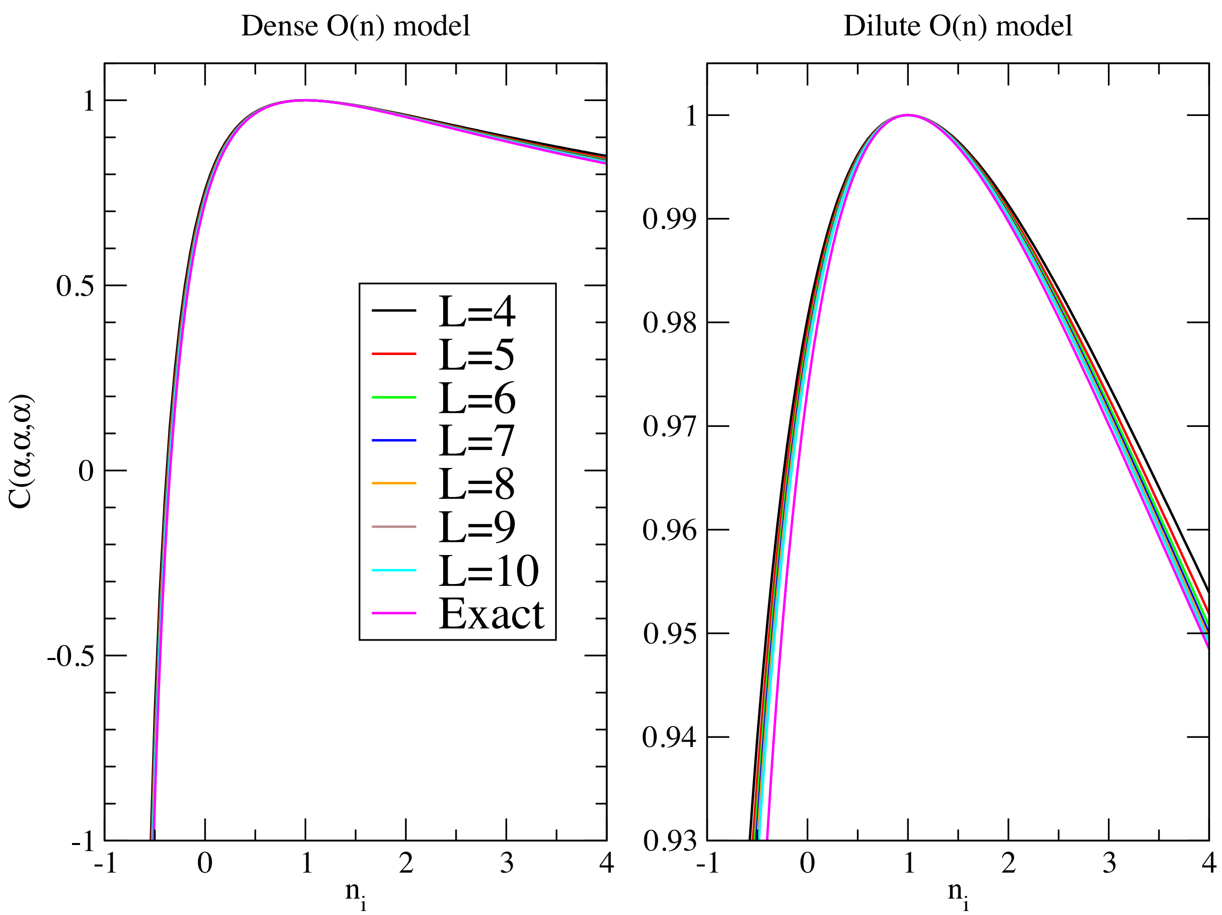
Define boundary conditions to impose weights n_1, n_3

Iterate TM action to project on ground states

Insert operator that gives weight n_2 to loops encircling the origin.

M/L up to 40

■ Some simple results:



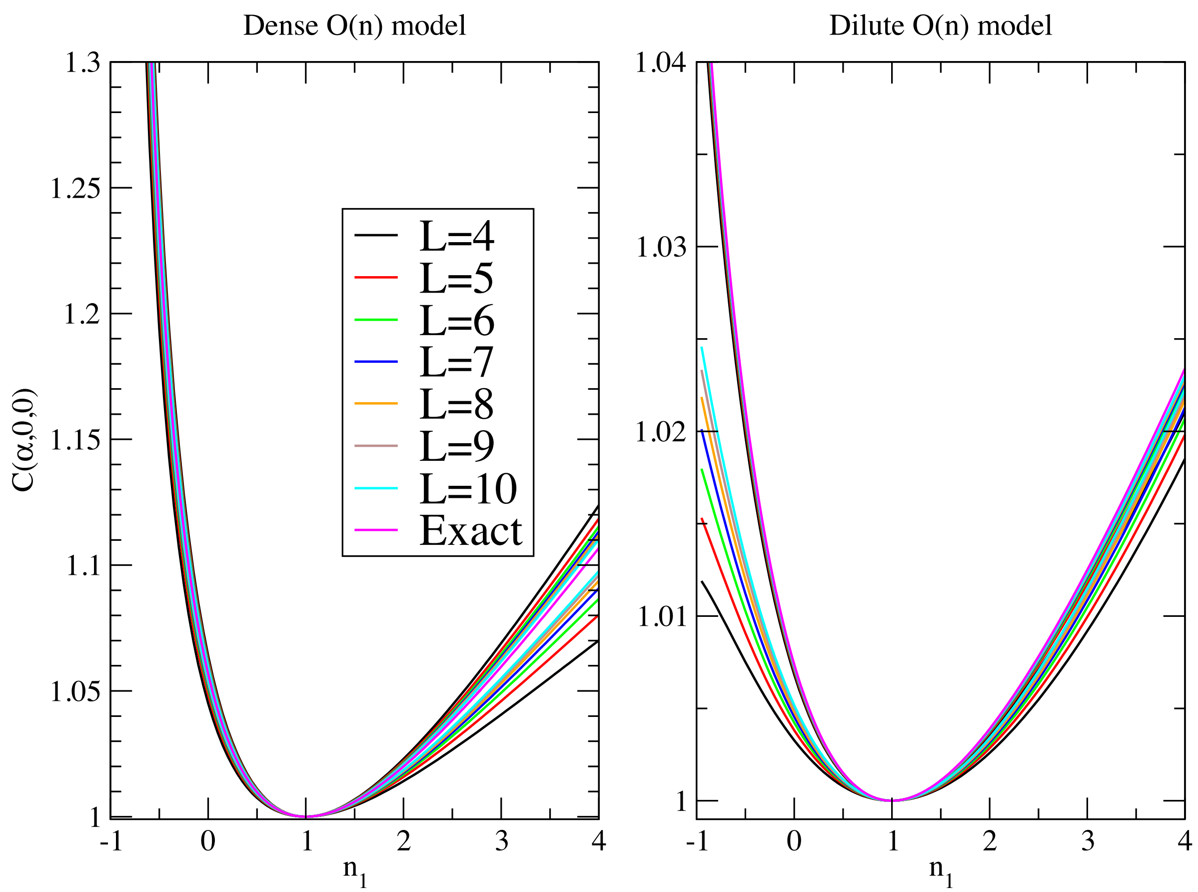
$\hat{C}(\hat{\alpha}, \hat{\alpha}, \hat{\alpha})$ as a function of $n_1 = n_2 = n_3$ in the dense and dilute O(n) model with $n = 1$.

- Excellent agreement
- Extends to $n_i > 2$, that is imaginary Liouville momentum

- Pole at $n_i = -1$ perfectly reproduced

- Some more subtle results: $\hat{C}(\hat{\alpha}_1, 0, \hat{\alpha}_3)$ is indeed non zero even when the charges are different.

Even $\hat{C}(\hat{\alpha}_1, 0, 0)$ is non zero and non trivial

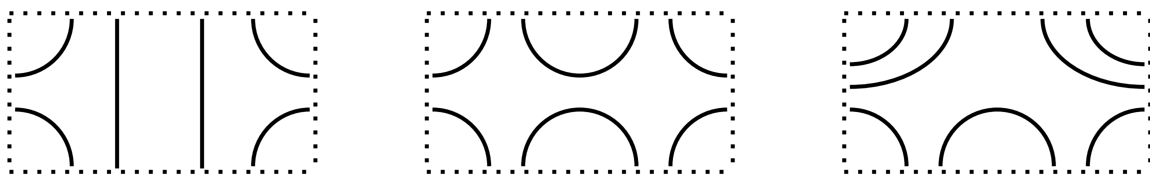


$\hat{C}(\hat{\alpha}_1, 0, 0)$ and $\hat{C}(0, \hat{\alpha}_1, 0)$ as functions of n_1 in the dense and dilute $\mathrm{O}(n)$ model with $n = 1$. The two positions of the vertex operator $V_{\hat{\alpha}_1}$—at one extremity or in the middle of the cylinder—give different microscopic results, but their $L \to \infty$ limits agree with the same analytical formula.

- Earlier result by [Delfino, Santachiara, Viti] verified the value of coupling constant in the special case $e_i = \frac{1}{2}, \quad n_i = 0$

But found agreement only up to a factor $\sqrt{2}$

This is a consequence of the fact that the relevant module for TL on the cylinder splits into two isomorphic submodules



(diagrams of odd and even rank are not connected by the algebra)

- Identifying the charge via $n_i \equiv 2\cos \pi e_i$

fixes it only up to an integer. Structure constants for the corresponding solutions are found by excited states in the TM.

- Case c=1 is particularly interesting: $C(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3) = \exp[Q(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)]$

$$Q(x,y,z) = \int_0^1 \frac{d\beta}{(-\ln\beta)(1-\beta)^2} \Bigg[2 + \sum_{\epsilon=\pm 1} (\beta^{\epsilon x/2} + \beta^{\epsilon y/2} + \beta^{\epsilon z/2}) - \sum_{\epsilon_x, \epsilon_y, \epsilon_z = \pm 1} \beta^{(\epsilon_x x + \epsilon_y y + \epsilon_z z)/4} \Bigg]$$

Note this is close to but not the same as the theory of [Runkel Watts], which is obtained as the limit of minimal models when $c \to 1$:

the RW theory three point function has singularities at degenerate conformal weights $\Delta = \frac{n^2}{4}$ which don't appear in the loop model.

- It does not seem that the C functions can be obtained from XXZ chain and Bethe ansatz. Among other things, this is because the loop combinatorics requires treating the q parameter in XXZ as a formal, self-conjugate parameter.

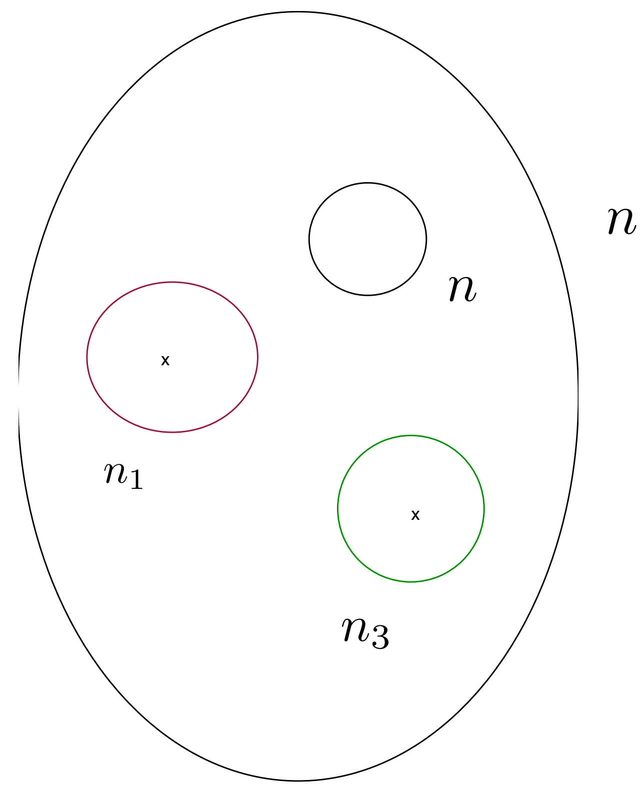
- When c=1, q=1, we have a better chance. But only the case $\hat{\alpha}_1 + \hat{\alpha}_2 = \hat{\alpha}_3$ is local for XXX, and in fact, $C(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_1 + \hat{\alpha}_2) = 1$ which is just a free boson result.

Some applications to loop models

- Consider $\hat{C}(\hat{\alpha}_1, 0, \hat{\alpha}_3)$. Even if $n_2 = n$ the weight of loops encircling points 1 or 3 still depends on whether they encircle point 2 or not. While associated the corresponding operator has weight zero, it is not trivial, and behaves like a marking operator (reminiscent of SLE in the boundary case)

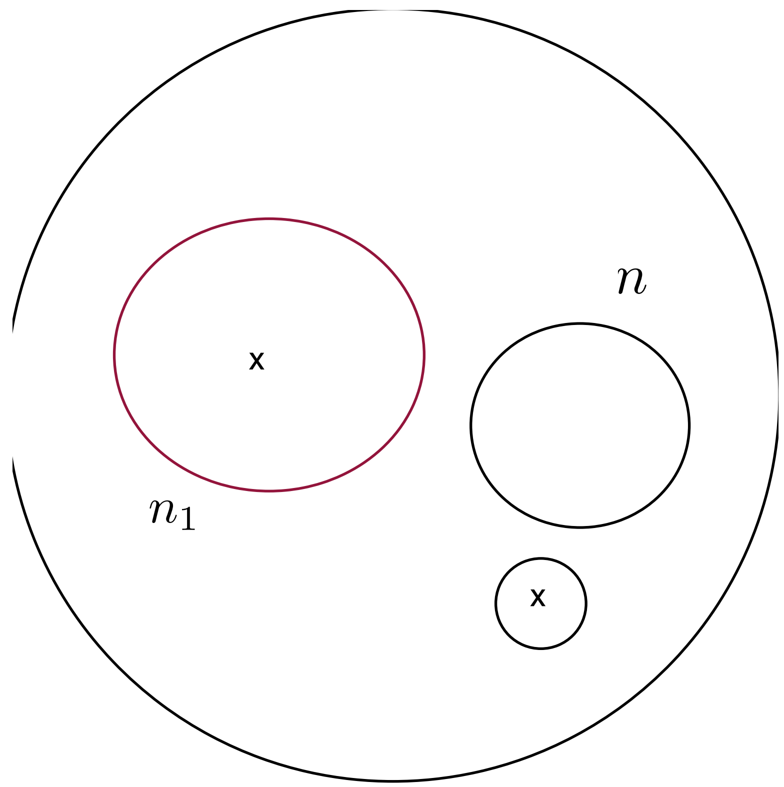
  Sending point 2 to infinity then leads to

$$\frac{Z_{n_1,n_3}}{\sqrt{Z_{n_1,n_1} Z_{n_3,n_3}}} = \hat{C}(\hat{\alpha}_1, 0, \hat{\alpha}_3)$$

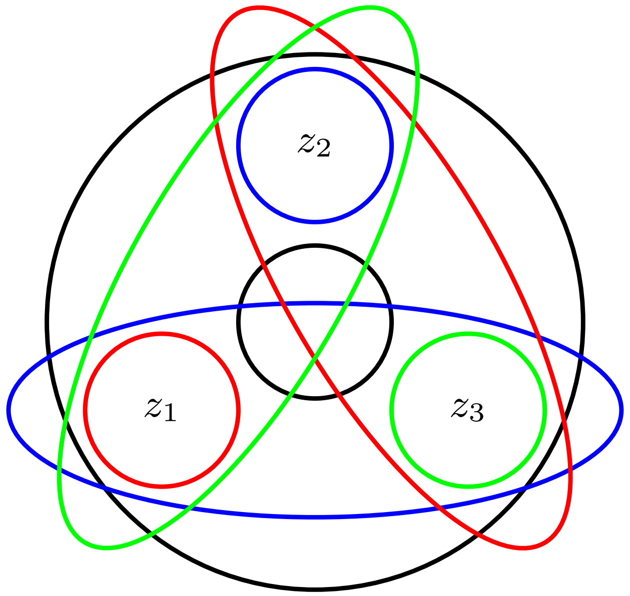


Consider $\hat{C}(\hat{\alpha}_1, 0, 0)$. Even in this case, combinatorics depends on the position of points 2 and 3.

$$\frac{Z_{n_1,n}}{\sqrt{Z_{n_1,n_1}Z}} = \hat{C}(\hat{\alpha}_1, 0, 0)$$



The geometrical meaning of fusion: as 2 gets close to 3, green and blue loops get pinched, and stop contributing to partition function. What's left are only red loops, i.e. an operator with charge $\hat{\alpha}_1$ at 2=3



$$V_{\hat{\alpha}_2} V_{\hat{\alpha}_3} \sim \int \hat{C}(\hat{\alpha}_1, \hat{\alpha}, \hat{\alpha}_3) V_{\hat{\alpha}_1}$$

Liouville c<1 as a CFT?

- In ordinary Liouville, crossing symmetry of four point function and modular invariance of one point function have been checked. The two objects involve conformal blocks which are a priori different, but between which relations are known to exist [Poghossian 09, Hadasz et al. 09].

- A similar check has been carried out recently by [Ribault Santachiara] for Liouville c<1.

- This involves the following ingredients: write generally

$$\langle V_\alpha \rangle = \mathrm{Tr}\left(q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} \right) = \sum_\Delta C^{\Delta}_{\Delta_\alpha, \Delta} |q|^{2\Delta - c/12} |\mathcal{F}^\Delta_\alpha(q)|^2$$

with the torus conformal blocks

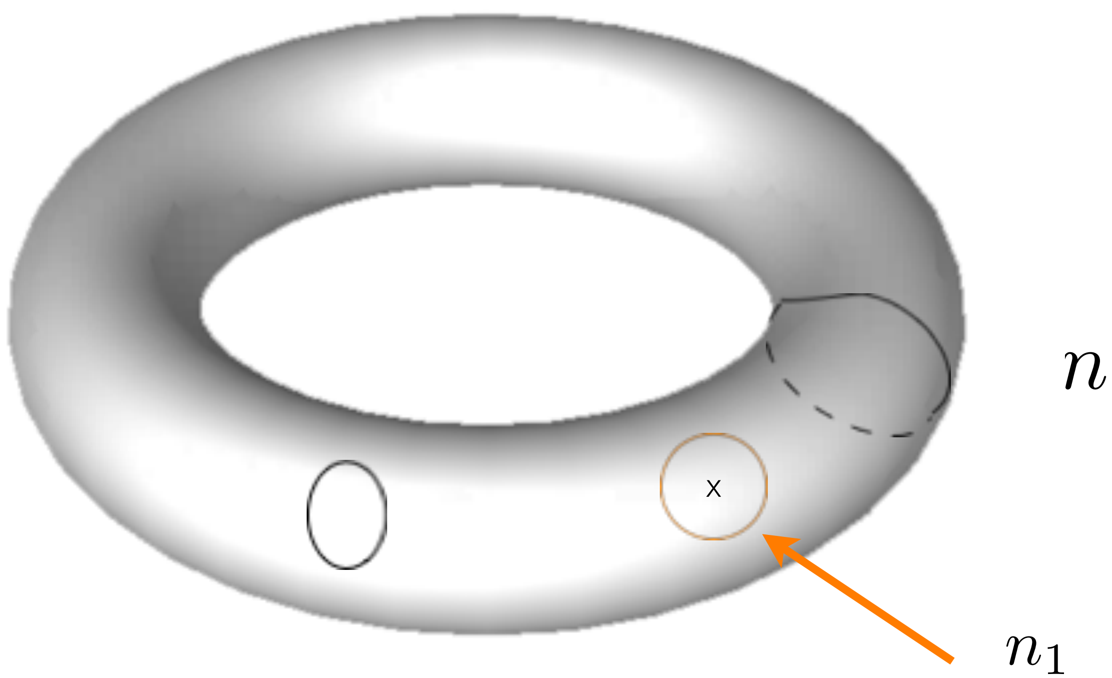
$$\mathcal{F}^\Delta_\alpha(q) \equiv \frac{1}{\langle \Delta | V_\alpha | \Delta \rangle} \left(\langle \Delta | V_\alpha | \Delta \rangle + \frac{\langle \Delta | L_1 V_\alpha L_{-1} | \Delta \rangle}{\langle \Delta | L_1 L_{-1} | \Delta \rangle} q + \dots \right) =$$
$$= 1 + \frac{2\Delta + \Delta(\alpha)^2 - \Delta(\alpha)}{2\Delta} q + \dots$$

- Defining $\mathcal{F}_\alpha^{(\Delta)}(q) \equiv \frac{q^{1/24}}{\eta(\tau)}\mathcal{H}_\alpha^{(\Delta)}(q)$

  and $\mathcal{H}_\alpha^{(\Delta)}(q) = 1 + \sum_{L=1}^{\infty} H_L(\Delta) q^L$ efficient recursion relations are known for the $H_L(\Delta)$

  Since the three point functions are known exactly, an expansion of the one point functions in powers of q follows if one knows the spectrum

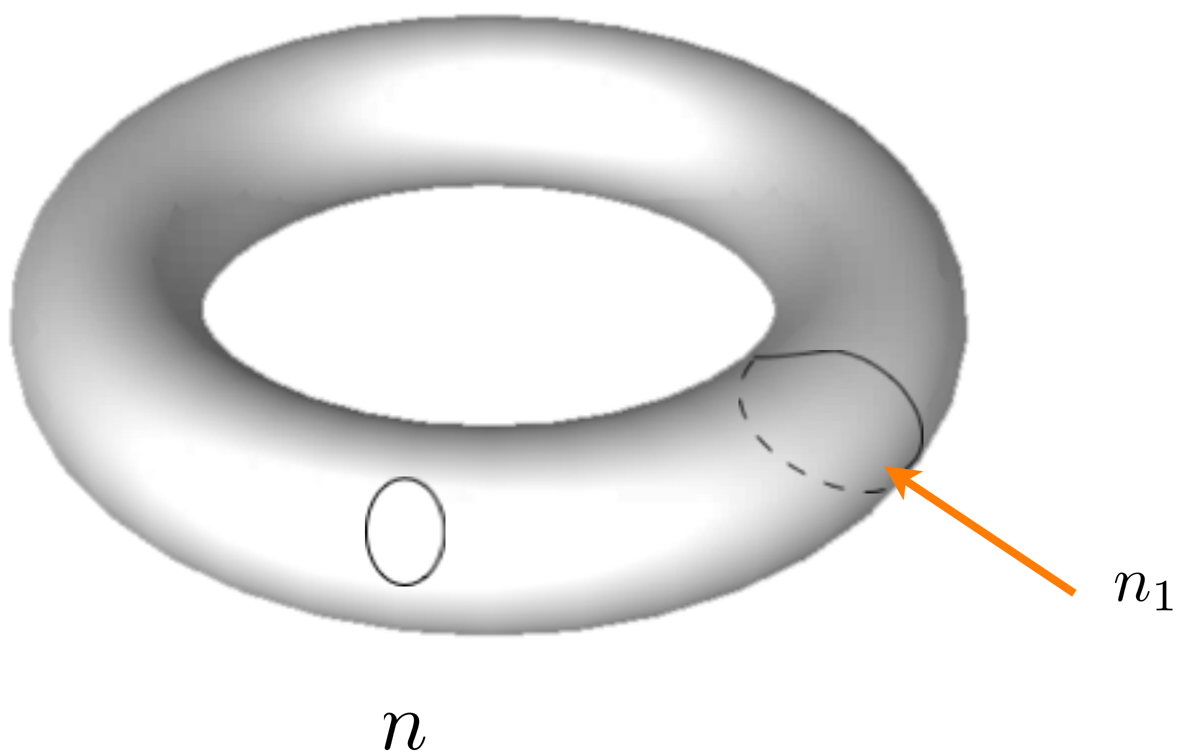
- We can meanwhile determine numerically what we believe is the one point function in the loop model



So far… it doesn't seem to work beyond the long torus (cylinder) limit.

The alternative is to study the four point function in loop models, which doesn't seem to work either. In both cases there are ambiguities : eg for the four point function, how to define the weights of loops encircling some points and not others. And for the torus, what to do with non-contractible loops, that is, roughly, what's the spectrum of the 'physical theory' if any. Note that Liouville-like theories have only scalar ($\Delta = \bar{\Delta}$) primary fields, in contrast with the 'natural' loop model … Work in progress.

Note:



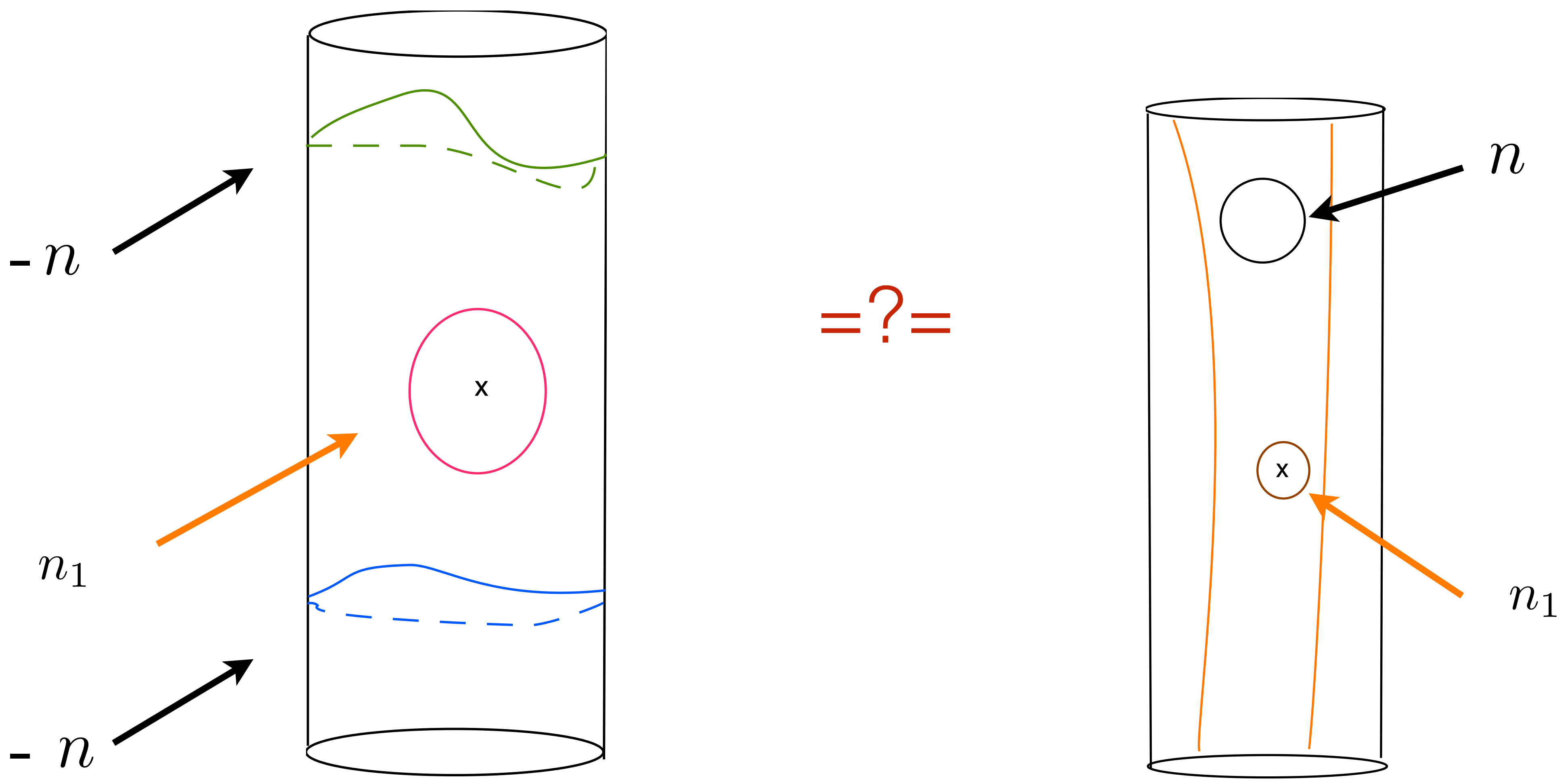
$$\int Z_T(n, n_1) d(\arccos n_1) \propto \frac{1}{\sqrt{\mathrm{Im}\tau}} \frac{1}{\eta\bar{\eta}}$$

the non compact free boson partition function

contractible (resp. non) loops get weight n (resp. n_1)

Other aspects

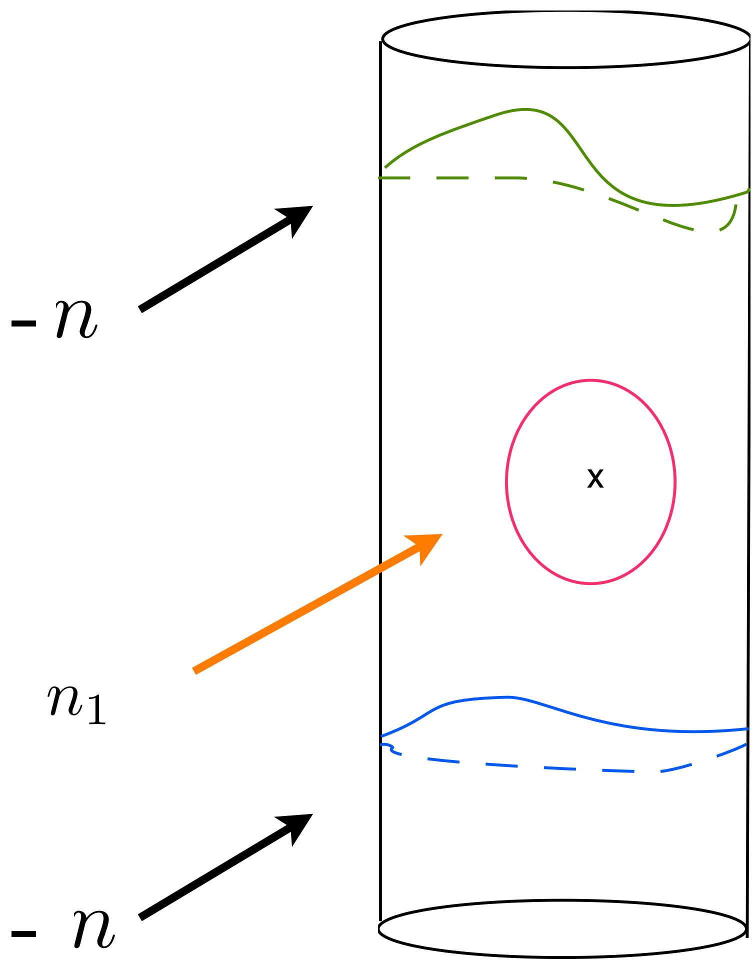
- The poles of the three point couplings occur exactly when the conformal weights of the vertex operators coincide with those of magnetic operators



$$\Delta = \frac{1}{4}\left(\frac{e}{\sqrt{g}} + m\sqrt{g}\right)^2 - \frac{(1-g)^2}{4g}, \quad e \sim gm$$

$$n = -2\cos \pi g : e_i = g \to n_i = -n$$

The quantity involving only loops

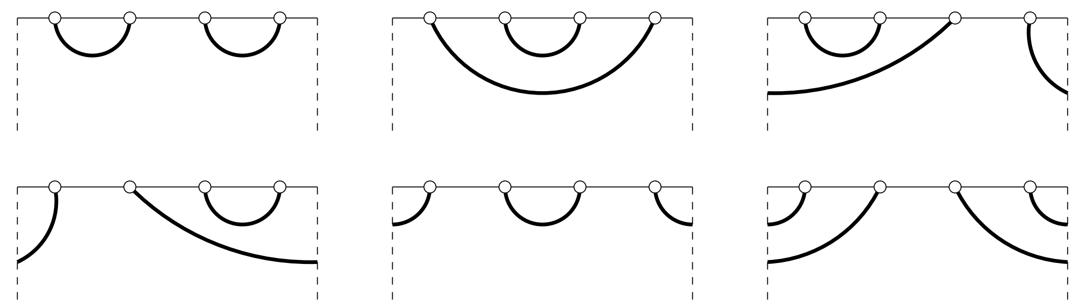


truly has a pole because the in and out states have zero norm square

This can be seen on the lattice thanks to representation theory of the (agumented) affine Temperley Lieb algebra.

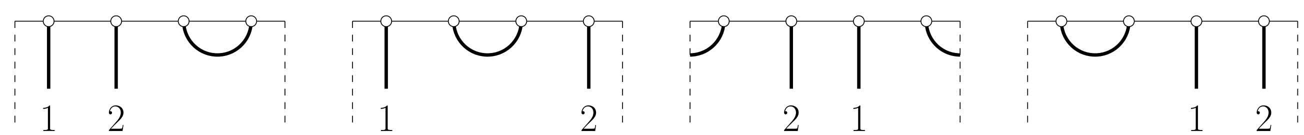
A good example is provided by the module with zero through lines $\mathcal{W}_{0,\mathfrak{q}^2}$

on four strands



which contains a submodule isomorphic to

$\mathcal{W}_{1,1}$



States in $\mathcal{W}_{0,\mathfrak{q}^2}$ and in the image of $\mathcal{W}_{1,1}$ are orthogonal on the lattice already.

From the continuum theory point of view, the vanishing of the norm can be understood since the conformal weights of the corresponding operators are formally given by $h_{0,m}$ (in Kac' conventions)

- The measures of three point couplings involving mixtures of electric and magnetic operators, or purely magnetic operators, give however finite results! So far we don't know how to obtain them analytically.

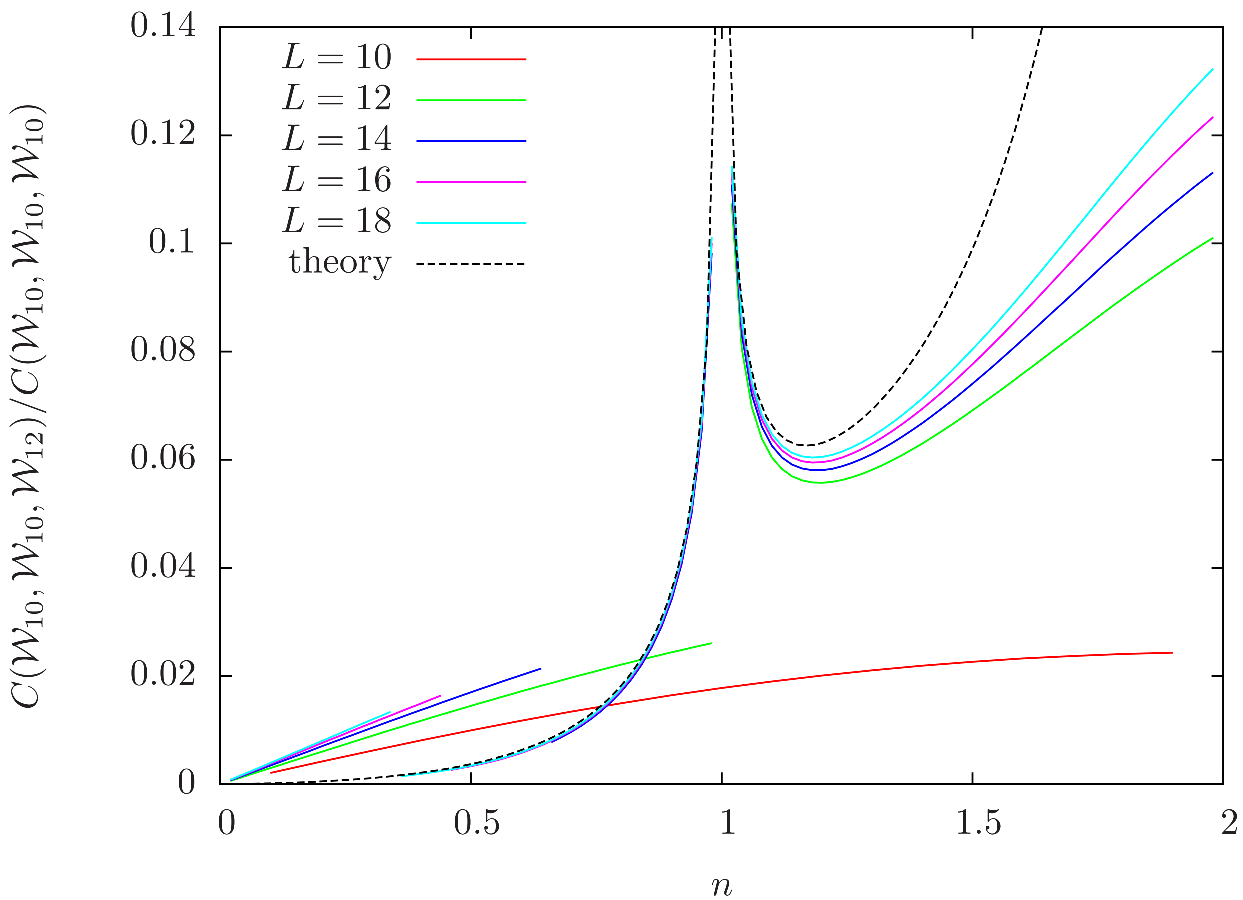
- We know some stuff however, for instance [Estienne, Ikhlef,2015] if

$$\Delta = \frac{1}{4}\left(\frac{e}{\sqrt{g}} + m\sqrt{g}\right)^2 - \frac{(1-g)^2}{4g}, \quad e \sim gm \quad \longleftrightarrow \quad \mathcal{W}_{me}$$

$$\frac{C(\mathcal{W}_{10},\mathcal{W}_{10},\mathcal{W}_{12})}{C(\mathcal{W}_{10},\mathcal{W}_{10},\mathcal{W}_{10})} = \left[\frac{\gamma(1-\rho)\gamma(-1-\rho)}{\gamma(-1+2\rho)\gamma(1+2\rho)\gamma(-1+\rho)\gamma(1+\rho)}\right]^{1/4} \times \left[\frac{\gamma^3(\rho/2)\gamma(-1+\rho/2)}{\gamma^3(-\rho/2)\gamma(-1-\rho/2)}\right]^{1/2},$$

where

$$\rho = \frac{1}{g} = \frac{1}{\hat{b}}, \ \gamma(x) \equiv \frac{\Gamma(x)}{\Gamma(1-x)}$$



Back to LCFTs

- The DOZZ formula and its c<1 extension are obtained by exploiting the fact that Φ_{12} & Φ_{21} are both degenerate at level two, and by setting the corresponding null vector to zero [Teschner].

$$\frac{C_{\mathrm{M}}(\alpha_1-\beta^{-1},\alpha_2,\alpha_3)}{C_{\mathrm{M}}(\alpha_1,\alpha_2,\alpha_3)}=\frac{\gamma(\beta^{-2}-(\alpha_1+\alpha_2-\alpha_3)\beta^{-1})\gamma(\beta^{-2}-(\alpha_3+\alpha_1-\alpha_2)\beta^{-1})}{\gamma(-(\alpha_2+\alpha_3-\alpha_1)\beta^{-1})\gamma(2-2\beta^{-2}+(\alpha_1+\alpha_2+\alpha_3)\beta^{-1})}\times$$

$$\times\,(\gamma(\beta^{-2}-2\alpha_1\beta^{-1})\gamma(2\beta^{-2}-2\alpha_1\beta^{-1})\times$$

$$\times\,\gamma(2\beta^{-2}-2\alpha_1\beta^{-1}-1)\gamma(3\beta^{-2}-2\alpha_1\beta^{-1}-1))^{-1/2}.$$

$$\frac{C_{\mathrm{M}}(\alpha_1+\beta,\alpha_2,\alpha_3)}{C_{\mathrm{M}}(\alpha_1,\alpha_2,\alpha_3)}=\frac{\gamma(\beta^2+(\alpha_1+\alpha_2-\alpha_3)\beta)\gamma(\beta^2+(\alpha_3+\alpha_1-\alpha_2)\beta)}{\gamma((\alpha_2+\alpha_3-\alpha_1)\beta)\gamma(2-2\beta^2-(\alpha_1+\alpha_2+\alpha_3)\beta)}\times$$

$$\times(\gamma(\beta^2+2\alpha_1\beta)\gamma(2\beta^2+2\alpha_1\beta)\gamma(-1+2\beta^2+2\alpha_1\beta)\gamma(-1+3\beta^2+2\alpha_1\beta))^{-1/2}.$$

Solution is unique for $\hat{b}^2$ not rational

The numerical checks of the formula suggest that in the loop model, this holds indeed.
The question is not clear from the point of view of LCFTs. Does the same hold for Φ_{n1} ?

How about $\phi_{21} \otimes \phi_{2,-1}$?

Conclusions

- We have to put together aspects of Liouville with the algebraic aspects of fusion etc. Can we build LCFTs as Liouville at c<1 + zero modes?
[Felstadt, Fuchs, Hwang,Semikhatov,Tipunin]

- A little mystery: discretizations of sine-Liouville (Witten's cigar theory), are known both in its ordinary regime c>2 [Ikhlef, Jacobsen,Saleur 2012] as well as its c<2 regime. So far we don't know any lattice model for ordinary Liouville.