CFTs and associative algebras

Physics is mathematical not because we know so much about the physical world, but because we know so little; it is only its mathematical properties that we can discover.

(Bertrand Russell)

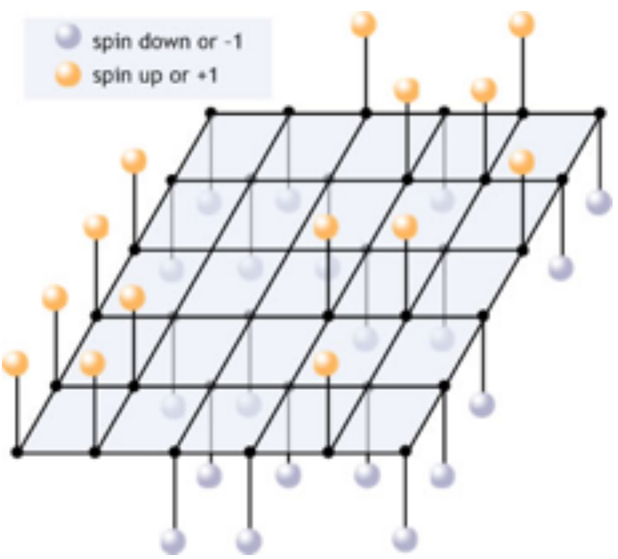
Campus Paris Saclay
Fondation de Coopération Scientifique

USC University of Southern California

European Research Council
Advanced Grant
erc

Why CFTheorists are interested in associative algebras

- 2 dim Critical statistical mechanics systems like the Ising model

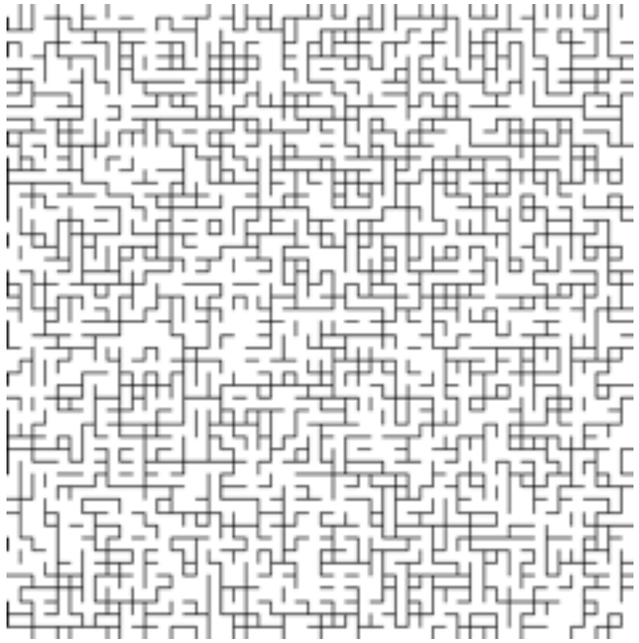


energy $$E(\{S_i\}) = -J \sum_{<i,j>} S_i S_j, \quad S_i = \pm 1$$

probability of a configuration $$P(\{S_i\}) = \frac{e^{-E(\{S_i\}/k_B T}}{Z}$$

partition function $$Z = \sum_{\{S_i\}} e^{-E(\{S_i\}/k_B T}$$

or percolation



probability of a configuration $$p^{\text{occupied edges}}(1-p)^{\text{unoccupied edges}}$$

partition function $$Z = 1$$

- have critical points where

correlation functions of local observables decay as power laws $\langle S_{i_1} S_{i_2} \rangle \sim \frac{1}{|r_{i_1} - r_{i_2}|^{2x}}$ scaling dimension

properties become universal and can be described by

conformal field theory

(which means in particular that correlation functions have nice properties under conformal transformations)

- For a physicist CFT involves pretty much two technical aspects

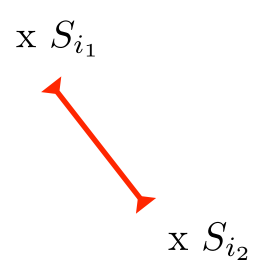
  - The Virasoro algebra and its representation theory

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m}$$

(infinite dimensional Lie algebra with central extension)

  - The conformal bootstrap and fusion

example:



$\sim \quad \epsilon \equiv \quad \text{energy } (= S_{i_1} S_{i_2},\ i_1, i_2 \text{ neighbors})$

this gets formalized in the language of vertex algebras and so called OPE algebras (Kapustin Orlov, Rosellen)

- **There are reasons why it is important to understand better these two aspects directly on the lattice - where, for instance, the conformal symmetry cannot be exact.**

one of these reasons is that we don't understand
much to Logarithmic CFT (non semi-simple cases)

In quantum field theory, unitarity is mandatory. It implies semi-simplicity, and, in many cases, allows full classification of Virasoro modules that can appear (e.g. c<1 classification, Friedan Qiu Shenker, Rocha Caridi, Feigin Fuchs)

In statistical mechanics, there is no such constraint. Percolation, Self-avoiding walks,disordered electronic systems all correspond to non-unitary CFTs. This translates into non semi-simple Virasoro representation theory. And Virasoro is wild (Germoni).

The hope is that we can understand what kind of algebraic properties to expect in the CFT from those we can investigate analytically/numerically on the lattice. That's the "associative algebraic approach to LCFT" (Read Saleur 2001)

Indeed an analysis of 2D critical statistical mechanics systems from a Hamiltonian point of view (transfer matrix) suggests that (in simple models like Ising or percolation)

$$e_i \sim T(z) + \bar{T}(\bar{z})$$

(Koo Saleur 93)

the elementary Temperley-Lieb generator

the stress energy tensor

$$[e_i, e_{i+1}] \sim T(z) - \overline{T}(\overline{z})$$

The dream is to define and understand a sort of limit process (thermodynamic+continuum limit) where the (enveloping algebra of the)Virasoro algebra can be realized in terms of TL generators acting on an infinite statistical mechanics system.

there is a lot to say about this
little of it is rigorous
quite a bit of it is numerical

….but we know how it should work

The open case

- In CFT, there are really two Virasoro algebras $L_n, \ \bar{L}_n$. That's because physical fields $\Phi(z,\bar{z})$ are non chiral.

- There is however a situation where physical fields are chiral, the so called Boundary CFT (BCFT) (Cardy).

this should correspond to the ordinary TL algebra $\mathsf{TL}_N(m)$

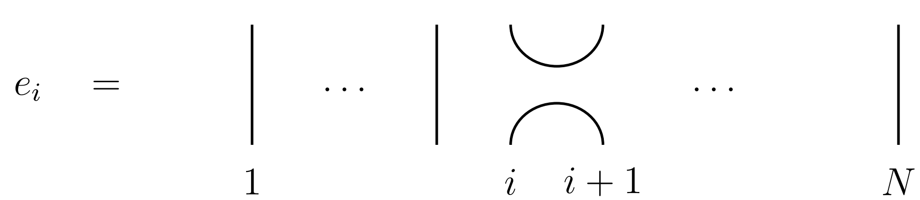


FIGURE 2. The diagrammatic representation of e_i.

$1 \leq j \leq N-1$

$$
\begin{aligned}
e_j^2 &= m e_j, \\
e_j e_{j\pm 1} e_j &= e_j, \\
e_j e_k &= e_k e_j \qquad (j \neq k, \ k \pm 1).
\end{aligned}
$$

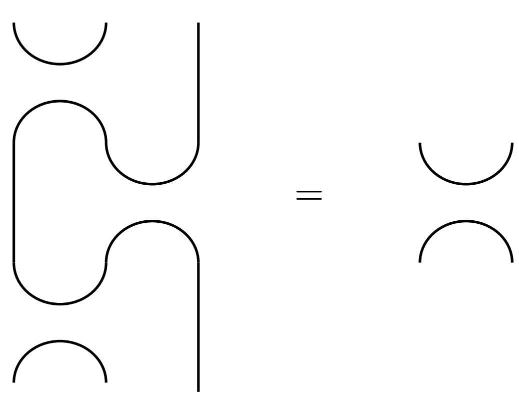


FIGURE 3. The diagrammatic version of the relation $e_i e_{i+1} e_i = e_i$.

- Standard modules $\mathcal{W}_j[N]$ have dimension $d_j[N] = \begin{pmatrix} N \\ \frac{N}{2}+j \end{pmatrix} - \begin{pmatrix} N \\ \frac{N}{2}+j+1 \end{pmatrix}$

They are irreducible for $\mathfrak{q}$ generic (where $m = \mathfrak{q} + \mathfrak{q}^{-1}$)

- A thing we know rigorously: set $\mathfrak{q} = e^{i\pi/x}$ and consider the Virasoro algebra for

$$c = 1 - \frac{6}{x(x-1)}$$

Kac formula

$$h_{rs} = \frac{[xr-(x-1)s]^2-1}{4x(x-1)}$$

For these values of conformal weight, Verma module $\mathcal{V}_h$ admits one singular vector at level $h_{rs}+rs$. The Kac module $\mathcal{K}_{r,s} \equiv \mathcal{V}_{h_{r,s}}/\mathcal{V}_{h_{r,-s}}$ is irreducible

now introduce lattice Hamiltonian

$$H = -\sum_{i=1}^{N-1} e_i$$

it turns out that

$$\mathrm{Lim}_{N\to\infty}\mathrm{Tr}\ e^{-\beta(H-Ne_0)} = \mathrm{Tr}\ q^{L_0-c/24} \qquad q=e^{-\pi\beta/N}$$

trace is taken over any $\mathcal{W}_j[N]$

trace is taken over Kac module $\mathcal{K}_{1,1+2j} = \mathcal{V}_{h_{1,1+2j}}/\mathcal{V}_{h_{1,-1-2j}}$

(and one has in fact $\mathrm{Tr}\ q^{L_0-c/24} = q^{-c/24}\dfrac{q^{h_{1,1+2j}} - q^{h_{1,-1-2j}}}{P(q)}$)

the question is, in what sense is the Kac module the "limit" of the TL modules?

- Note: there is a better dictionary involving all Kac modules, and the blob (one boundary TL) algebra Gainutdinov Jacobsen Saleur Vasseur 2013

- There are many ways ot think of this problem:

One is to try to build the whole Virasoro action within the TL modules
Koo Saleur 1994

Virasoro generator in the open case:

$$L_n^{(N)} = \frac{N}{\pi}\left[-\frac{1}{v_F}\sum_{i=1}^{N-1}(e_i - e_0)\cos\frac{ni\pi}{N} + \frac{1}{v_F^2}\sum_{i=1}^{N-2}[e_i, e_{i+1}]\sin\frac{ni\pi}{L}\right] + \frac{c}{24}\delta_{n,0}$$

There is in fact an infinity of such "approximations" which close only in the thermodynamic limit, and when restricting to "scaling states"

This remains a long (and dirty) story

another is to study fusion Read Saleur 2001

Definition 2.2.1 ([21, 29]). *Let M_1 and M_2 be two modules over TL_{N_1} and TL_{N_2} respectively. Then, the tensor product $M_1 \otimes M_2$ is a module over the product $\mathsf{TL}_{N_1} \otimes \mathsf{TL}_{N_2}$ of the two algebras. Using the standard embedding, we consider this product of algebras as a subalgebra in TL_N, for $N = N_1 + N_2$. The fusion (bi-)functor*

$$\times_f : \quad \mathsf{C}_{N_1} \times \mathsf{C}_{N_2} \to \mathsf{C}_{N_1+N_2} \tag{2.4}$$

on two modules M_1 and M_2 is then defined as the module induced from this subalgebra, i.e.,

$$M_1 \times_f M_2 = \mathsf{TL}_N \otimes_{\left(\mathsf{TL}_{N_1} \otimes \mathsf{TL}_{N_2}\right)} M_1 \otimes M_2 \ , \tag{2.5}$$

where we used the balanced tensor product over $\mathsf{TL}_{N_1} \otimes \mathsf{TL}_{N_2}$.

Straightforward results in the generic case.

$$\mathcal{W}_{j_1}[N_1] \times_f \mathcal{W}_{j_2}[N_2] = \bigoplus_{|j_1-j_2|}^{j_1+j_2} \mathcal{W}_j[N_1+N_2]$$

SU(2)q, Schur-Weyl

matches the expected result in the CFT

$$\phi_{1,1+2j_1}.\phi_{1,1+2j_2} \sim \sum_{|j_1-j_2|}^{j_1+j_2} \phi_{1,1+2j}$$

Complex and fascinating results when $\mathfrak{q}$ is a root of unity

physical models seem to always involve projective modules (glueing of two standard modules if $j \geq \frac{p}{2}$)

$$\mathcal{W}_j[N] \ = \ \mathcal{X}_j[N] \longrightarrow \mathcal{X}_{j+p-s}[N]$$

$$\mathfrak{q} \ = \ e^{\mathrm{i}\pi/p}$$

$$\mathcal{P}_j[N] \quad = \quad \begin{tikzcd} & \mathcal{X}_j[N] \arrow[dl] \arrow[dr] & \\ \mathcal{X}_{j-s}[N] \arrow[dr] & & \mathcal{X}_{j+p-s}[N] \arrow[dl] \\ & \mathcal{X}_j[N] & \end{tikzcd}$$

$$s \ \equiv \ s(j) \ = \ (2j+1) \mod p$$

$$s(j) \neq 0$$

their continuum limit is given by Virasoro staggered modules
(extension of two highest weight modules, with L_0 non diagonalizable

$$\mathcal{P}_{1,2j+1} : \quad \begin{tikzcd} & h_{1,2j+1} \arrow[dl] \arrow[dr] & \\ h_{1,1+2(j-s)} \arrow[dr] & & h_{1,1+2(j+p-s)} \arrow[dl] \\ & h_{1,2j+1} & \end{tikzcd} \qquad \text{for} \quad j \geq \frac{p}{2},$$

TL fusion of projectives “matches” fusion of staggered Virasoro modules in LCFT!

Kytola, Ridout, St Aubin, Kausch, Gaberdiel, Nahm, Pearce, Rasmussen, Belletete, Jacobsen, Gainutdinov, Read, Saleur [2007-2016]

precise categorical equivalence   Gainutdinov Saleur 2016

5.2.2. Conjecture. *For $1 \leqslant s_1, s_2 \leqslant p-1$ and $r_1, r_2 \geqslant 1$, the fusion of two staggered modules over the Virasoro algebra $\mathcal{V}(p-1,p)$ with central charge $c_{p-1,p}$ is*

$$\mathcal{P}_{1,r_1p+s_1} \times_f \mathcal{P}_{1,r_2p+s_2} = 4 \bigoplus_{\substack{r=|r_1-r_2|-1\\ \text{step}=2}}^{r_1+r_2+1}{}'' \bigoplus_{\substack{s=2p-s_1-s_2+1\\ \text{step}=2}}^{p-\gamma_2} \mathcal{P}_{1,(r+1)p-s}$$
$$\oplus\, 2 \bigoplus_{\substack{r=|r_1-r_2|+1\\ \text{step}=2}}^{r_1+r_2-1} \left(\bigoplus_{\substack{s=|s_1-s_2|+1\\ \text{step}=2}}^{\substack{\min(s_1+s_2-1,\\ 2p-s_1-s_2-1)}} \mathcal{P}_{1,(r+1)p-s} \oplus 2 \bigoplus_{\substack{s=s_1+s_2+1\\ \text{step}=2}}^{p-\gamma_2} \mathcal{P}_{1,(r+1)p-s} \right)$$
$$\oplus\, 2 \bigoplus_{\substack{r=|r_1-r_2|\\ \text{step}=2}}^{r_1+r_2}{}' \left(\bigoplus_{\substack{s=|p-s_1-s_2|+1\\ \text{step}=2}}^{\substack{\min(p-s_1+s_2-1,\\ p+s_1-s_2-1)}} \mathcal{P}_{1,(r+1)p-s} \oplus 2 \bigoplus_{\substack{s=\min(p-s_1+s_2+1,\\ p+s_1-s_2+1)}}^{p-\gamma_1} \mathcal{P}_{1,(r+1)p-s} \right).$$

Gainutdinov Vasseur 2014

In general: lots of progress in study of boundary LCFTs using this approach

Fusion in affine TL

Gaynutdinov, Jacobsen, Saleur, 2016

- We now go back to the bulk (non-boundary) case. This should correspond to a TL algebra acting on a periodic system, the affine TL (Martin-Saleur 93, Jones 94, Green 98, Erdmann Green 99)

3.1.1. *Definition I: generators and relations. The affine Temperley–Lieb (aTL) algebra* $\mathsf{T}^a_N(m)$ is an associative algebra over $\mathbb{C}$ generated by u, u^{-1} , and e_j, with $j \in \mathbb{Z}/N\mathbb{Z}$, satisfying the defining relations

$$
\begin{aligned}
e_j^2 &= m e_j, \\
e_j e_{j\pm1} e_j &= e_j, \qquad\qquad\qquad (3.1)\\
e_j e_k &= e_k e_j \qquad (j \neq k,\ k \pm 1),
\end{aligned}
$$

which are the standard TL relations but defined for the indices modulo N, and

$$
\begin{aligned}
u e_j u^{-1} &= e_{j+1}, \\
u^2 e_{N-1} &= e_1 \dots e_{N-1}, \qquad (3.2)
\end{aligned}
$$

where the indices $j = 1, \dots, N$ are again interpreted modulo N.

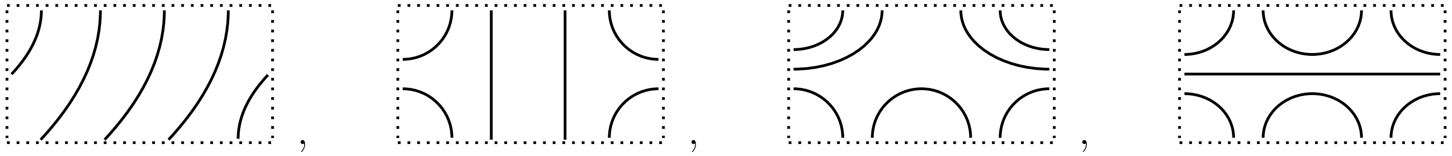
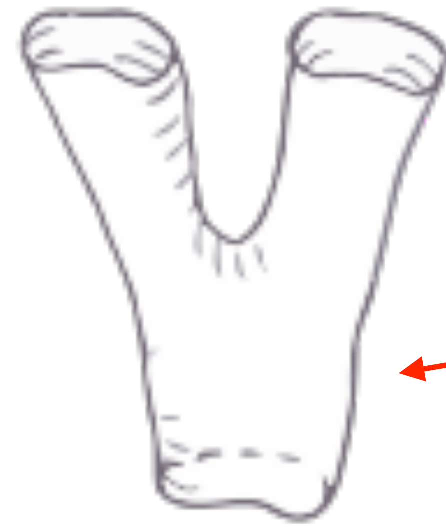


FIGURE 4. Examples of affine diagrams for $N = 4$, with the left and right sides of the framing rectangle identified. The first diagram represents the translation generator u while the second diagram is for the generator $e_4 \in \mathsf{T}^a_4(m)$. The third and fourth ones are examples of $j = 0$ diagrams.

- Note that diagrams in this algebra allow winding of through lines around the annulus any number of times, and different windings result in independant algebra elements. Moreover, in the ideal of zero through lines, any number of non-contractible loops is allowed. The algebra is thus infinite dimensional.

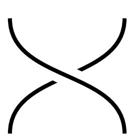
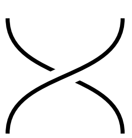
- Fusion in this affine case requires glueing two cylinders. How do we do this without cutting them open?

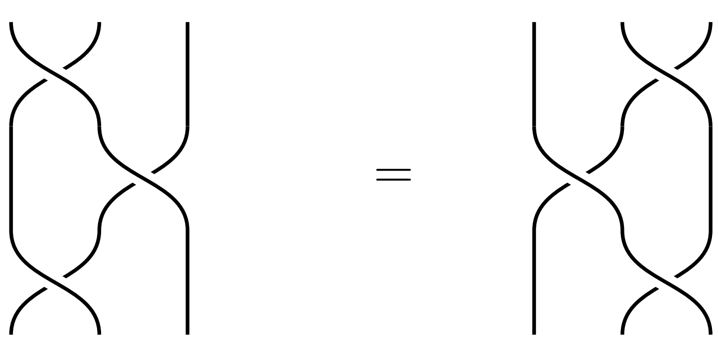


down there, the legs have disappeared!

- Affine braid group

$$g_i^{\pm 1} = \pm \mathrm{i}(\mathfrak{q}^{\pm\frac{1}{2}}\mathbf{1} - \mathfrak{q}^{\mp\frac{1}{2}} e_i)$$

$g_i \quad =$  $\qquad g_i^{-1} \quad =$ 



$(g_i g_{i\pm1} g_i = g_{i\pm1} g_i g_{i\pm1})$

The trick Gaynutdinov Saleur 2016

we can embed the product of two affine TL algebras, $\mathsf{T}^a_{N_1}$ and $\mathsf{T}^a_{N_2}$, into T^a_N with $N = N_1 + N_2$. Let us denote the generators in the ith algebra as $u^{(i)}$ and $e^{(i)}_j$, with $i = 1, 2$, and use standard notations for the generators in the "big" algebra T^a_N. We first define the map on the TL generators $e^{(i)}_j$, where $j \neq 0$, in the standard way

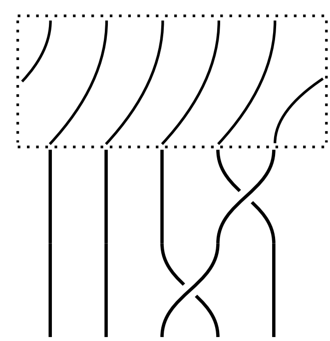
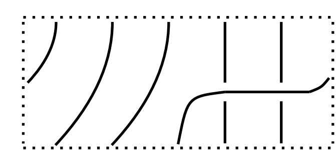
$$e^{(1)}_j \mapsto e_j, \qquad e^{(2)}_k \mapsto e_{N_1+k}, \qquad 1 \leq j \leq N_1 - 1, \quad 1 \leq k \leq N_2 - 1. \tag{3.7}$$

The translation generators $u^{(1)}$ and $u^{(2)}$ are mapped as (recall, we set $N = N_1 + N_2$)

$$u^{(1)} \mapsto u\, g^{-1}_{N-1} \dots g^{-1}_{N_1}, \qquad u^{(2)} \mapsto g_{N_1} \dots g_1 u. \tag{3.8}$$

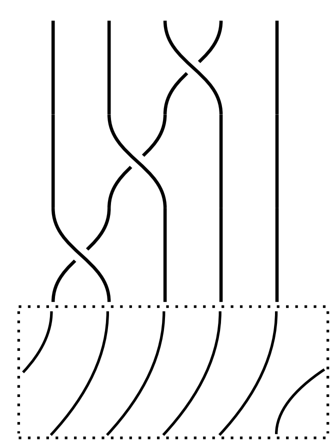
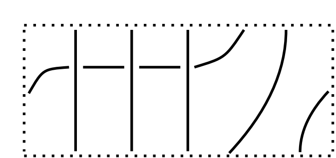
in terms of diagrams:

above

(3.9) $u^{(1)} \mapsto$  $=$ 

where we assumed that $N_1 = 3$ and $N_2 = 2$, and for the second translation $u^{(2)}$ we have the diagram

under

(3.10) $u^{(2)} \mapsto$  $=$ 

- One can check that:

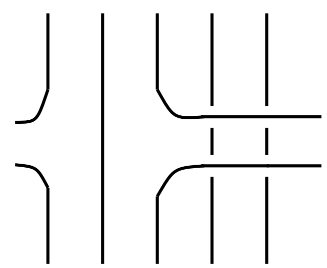
$$u^{(1)}u^{(2)} = u^{(2)}u^{(1)}$$

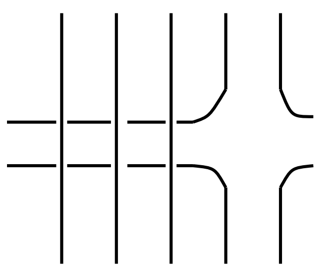
$$\left(u^{(1)}\right)^2 e_{N_1-1} = e_1 \ldots e_{N_1-1}, \qquad \left(u^{(2)}\right)^2 e_{L-1} = e_{N_1+1} \ldots e_{L-1}$$

- Next, we define the map on the periodic TL generators

$$\begin{aligned} e_0^{(1)} &\mapsto g_{N_1} \ldots g_{N-1}\, e_0\, g_{N-1}^{-1} \ldots g_{N_1}^{-1}, \\ e_0^{(2)} &\mapsto g_0^{-1} \ldots g_{N_1-1}^{-1}\, e_{N_1}\, g_{N_1-1} \ldots g_0 \end{aligned}$$

in terms of diagrams:

$e_0^{(1)} \quad =$ 

$e_0^{(2)} \quad =$ 

- One can check that:

$$\left(e_0^{(i)}\right)^2 = (\mathfrak{q} + \mathfrak{q}^{-1})e_0^{(i)}, \qquad i = 1, 2.$$

$$e_0^{(i)} = u^{(i)} e_{N_i-1}^{(i)} \left(u^{(i)}\right)^{-1} = \left(u^{(i)}\right)^{-1} e_1^{(i)} u^{(i)}, \qquad i = 1, 2,$$

$$e_0^{(i)} e_1^{(i)} e_0^{(i)} = e_0^{(i)}, \qquad e_1^{(i)} e_0^{(i)} e_1^{(i)} = e_1^{(i)},$$
$$e_0^{(i)} e_{N_i-1}^{(i)} e_0^{(i)} = e_0^{(i)}, \qquad e_{N_i-1}^{(i)} e_0^{(i)} e_{N_i-1}^{(i)} = e_{N_i-1}^{(i)},$$

$$e_0^{(1)} e_0^{(2)} = e_0^{(2)} e_0^{(1)}$$

this holds only because of the above/under pattern

- So now we can do fusion:

Definition 4.1. *Let M_1 and M_2 be two modules over $\mathsf{T}^a_{N_1}(m)$ and $\mathsf{T}^a_{N_2}(m)$ respectively. Then, the tensor product $M_1 \otimes M_2$ is a module over the product $\mathsf{T}^a_{N_1}(m) \otimes \mathsf{T}^a_{N_2}(m)$ of the two algebras. Using the embedding (3.24), we consider this product of algebras as a subalgebra in $\mathsf{T}^a_N(m)$, for $N = N_1 + N_2$. The (affine) fusion functor $\widehat{\times}_f$ on two modules M_1 and M_2 is then defined as the module induced from this subalgebra, i.e.*

$$M_1 \,\widehat{\times}_f\, M_2 = \mathsf{T}^a_N \otimes_{\left(\mathsf{T}^a_{N_1} \otimes \mathsf{T}^a_{N_2}\right)} M_1 \otimes M_2, \tag{4.1}$$

where we used the balanced tensor product over $\mathsf{T}^a_{N_1} \otimes \mathsf{T}^a_{N_2}$ and we abuse the notation by writing T^a_N instead of $\mathsf{T}^a_N(m)$.

- The results are a bit complicated. First, introduce standard modules $\mathcal{W}_{j,z}[N]$ (Martin-Saleur/Jones/Graham Lehrer)

Here 2j is the number of through lines as usual. z is a complex number whose role is to “unwind” through lines that go around the cylinder: whenever the 2j lines go arond clockwise we unwind them at the price of a factor 1/z;counterclockwise leads to a factor z instead. Finally, for j=0, non contractible loops are eliminated for a factor z+1/z

$$\hat{d}_j[N] \equiv \dim \mathcal{W}_{j,z}[N] = \binom{N}{\frac{N}{2}+j}$$

The method of "postulating" what we want has many advantages; they are the same as the advantages of theft over honest toil.   (Bertrand Russell)

we then have the conjectured results (based on direct calculations and Frobenius reciprocity)

$$\mathcal{W}_{j_1,z_1}[N_1] \mathbin{\widehat{\times}_f} \mathcal{W}_{j_2,z_2}[N_2] = \mathcal{W}_{j,z}[N_1+N_2]$$

no sum!

- For $j = j_1 + j_2$ and any values of j_1, j_2:

$$z_1 = (i\sqrt{\mathfrak{q}})^{-2j_2} z^{+1}, \qquad z_2 = (i\sqrt{\mathfrak{q}})^{+2j_1} z^{+1}$$

- For $j = j_1 - j_2$ and either $j = 0$ or $j_2 > 0$:

$$z_1 = (i\sqrt{\mathfrak{q}})^{+2j_2} z^{+1}, \qquad z_2 = (i\sqrt{\mathfrak{q}})^{-2j_1} z^{-1}$$

- For $j = j_2 - j_1$ and either $j = 0$ or $j_1 > 0$:

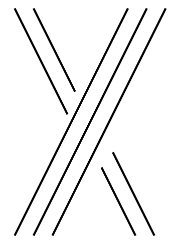
$$z_1 = (i\sqrt{\mathfrak{q}})^{+2j_2} z^{-1}, \qquad z_2 = (i\sqrt{\mathfrak{q}})^{-2j_1} z^{+1}$$

Gainutdinov Jacobsen Saleur 2016

otherwise fusion is zero

- this fusion is non-commutative, and associative

- this exists another fusion $\widehat{\times}_f^-$ obtained by switching over and under, and the two are related by braiding



:

$$M_1[N_1] \,\widehat{\times}_f\, M_2[N_2] \xrightarrow{\cong} M_2[N_2] \,\widehat{\times}_f^-\, M_1[N_1]$$

- a technical remark: it is well known how affine TL can be obtained as a quotient of affine Hecke. There is meanwhile a well known fusion in affine Hecke, Zelevinsky tensor product. The problem is, that this tensor product and the quotient to get affine TL are not, in general, compatible (so the result is "zero"). We have checked that, when it is compatible, our results are recovered.

there's room for a theorem!

CFT: the closed case

- Like in the open case, if we again take the Hamiltonian

$$H = -\sum_i e_i$$

together with the logarithm of the translation generator as the momentum P, we know that

$$\mathrm{Tr}\; e^{-\beta_R (H - N e_0)} e^{-i\beta_I P} \xrightarrow{N\to\infty} \mathrm{Tr}\, q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24}$$

trace taken over modules of ATL

trace taken over modules of $\mathrm{Vir}\times\overline{\mathrm{Vir}}$

where $q(\bar{q}) = \exp\left[-\frac{2\pi}{N}(\beta_R \pm i\beta_I)\right]$ (Cardy)

One finds, then that our fusion corresponds to glueing the right component of one field with the left component of the other field. Schematically:

Verma modules with Virasoro highest weight h,h'

$$\mathcal{W}_{j,z} \mapsto \mathrm{Vir}_h \times \overline{\mathrm{Vir}}_{h'}$$

where like before $\mathfrak{q} = e^{i\pi/x}$, $c = 1 - \frac{6}{x(x-1)}$ and h,h' are functions of j,z

Our Fusion now corresponds to

$$\left(\mathrm{Vir}_h \times \overline{\mathrm{Vir}}_{h'}\right) \ \widehat{\times}_f \ \left(\mathrm{Vir}_{h'} \times \overline{\mathrm{Vir}}_{h''}\right) = \left(\mathrm{Vir}_h \times \overline{\mathrm{Vir}}_{h''}\right)$$

the same conformal weight

This doesn't seem particularly useful

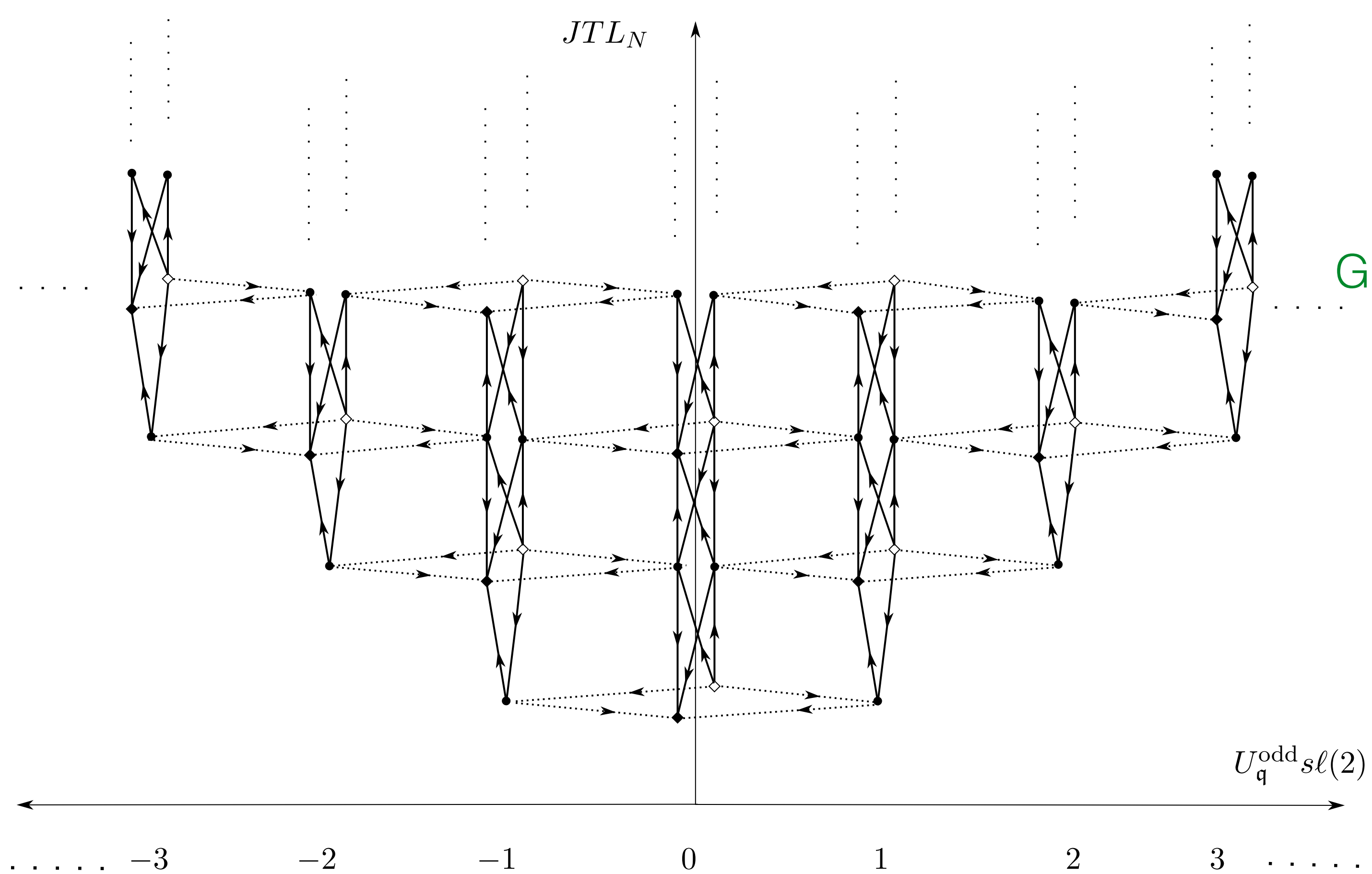
Conclusions

- Questions for mathematicians: Fusion in other algebras? Blob/boundary Temperley-Lieb/Temperley-Lieb type B,C? Other (better) ways to do fusion?

- Physicists have to do representation theory to understand in detail the relationship between lattice models and their conformal invariant limits. This is particularly crucial to make progress on logarithmic CFTs (non semi-simple VOAs) which play a role in the description of many systems of interest (in particular those involving disorder)

- Apart from modules and fusion, another hot topic is the understanding of lattice models "Hilbert spaces" as bimodules over ATL and its centralizer

(m=0,gl(1/1))



Gainutdinov, Read, Saleur 2013