Crossovers in impurity entanglement: analytical results

by H. Saleur (USC + Saclay)

Entanglement and quantum impurities: a well studied, and still lively, problem

Entanglement entropy in quantum impurity systems and systems with boundaries
Ian Affleck, Nicolas Laflorencie, Erik S. Sorensen
Comments: 40 pages and 22 figures. Review article for the special issue "Entanglement entropy in extended systems" in J. Phys. A
Journal-ref: J. Phys. A: Math. Theor. 42 504009 (2009)

Impurity entanglement entropy in Kondo systems from conformal field theory
Erik Eriksson, Henrik Johannesson
Comments: 4 pages, published version
Journal-ref: Phys. Rev. B 84, 041107(R) (2011)

An order parameter for impurity systems at quantum criticality
Abolfazl Bayat, Henrik Johannesson, Sougato Bose, Pasquale Sodano
Comments: 6 pages, 5 figures
Journal-ref: Nature Communications 5, 3784 (2014)

Fascinating also from a technical perspective, as it combines interactions with the non locality of entanglement

Infra-red expansion of entanglement entropy in the Interacting Resonant Level Model
Loïc Freton, Edouard Boulat, Hubert Saleur
Comments: 27 pages, 11 figures
Journal-ref: Nuclear Physics B 874 (2013) 279-311

Entanglement in quantum impurity problems is non perturbative
Hubert Saleur, Peter Schmitteckert, Romain Vasseur
Comments: 18 pages, 4 figures. v3: minor post-publication footnote added
Journal-ref: Phys. Rev. B 88, 085413 (2013)

Universal entanglement crossover of coupled quantum wires
Romain Vasseur, Jesper Lykke Jacobsen, Hubert Saleur
Comments: v2: to appear in PRL
Journal-ref: Phys. Rev. Lett. 112, 106601 (2014)

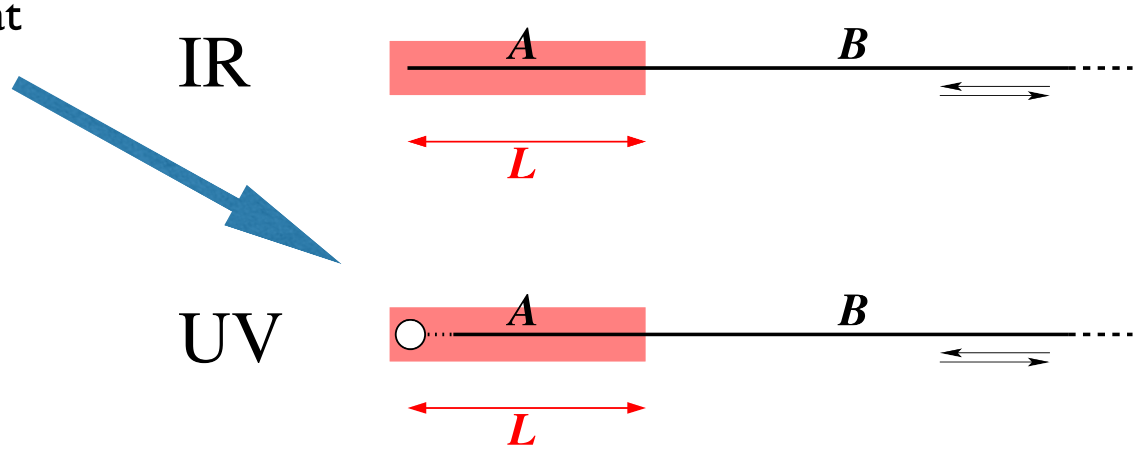
A few aspects…

The problem:

Entanglement crossovers in RG flows

in the context of quantum impurity problems. Some examples:

Impurity decouples at high energy

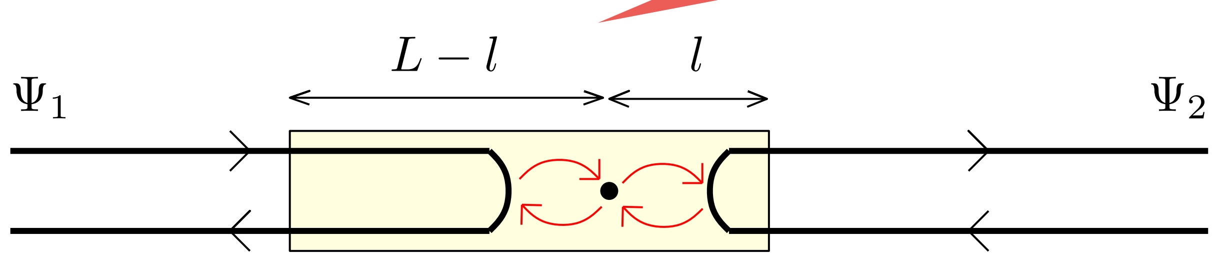


Kondo problem after reduction to the s channel

Affleck Laflorencie Sorensen 2006

or

Region A of total length L

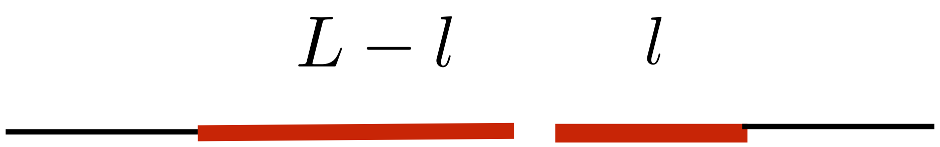


Tunneling through a quantum dot with dot/wires interactions, or with interactions within wire

Levine 2004, Eisler Peschel 2010

Healing at low energy

by a variety of tricks these problems are all technically related. To fix ideas let's take the second problem. In the UV the system is cut in half, and the entanglement of A with the rest is



$$S_{UV} = \frac{1}{6}\left(\ln\frac{2(L-l)}{\epsilon} + \ln\frac{2l}{\epsilon}\right) + \ln g_{UV}$$

using formulas for entanglement of boundary regions

In the IR it is healed

$$S_{IR} = \frac{1}{3}\ln\frac{L}{\epsilon} + \ln g_{IR}$$

Calabrese Cardy

we allowed for impurity (boundary)entropies (Affleck Ludwig) in the $O(1)$ g factors

We want to know how S varies in between

Entanglement is non perturbative

To fix ideas some more: take

$$\begin{aligned} H &= J \sum_{i=-\infty}^{\infty} \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z\right) \\ &+ (J' - J) \sum_{i=-1}^{0} \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z\right) \end{aligned}$$

J' is the perturbation (tunneling amplitude). Setting $h = 2\left(1 - \frac{1}{\pi} \arccos \Delta\right)$

Perturbation has dimension D=h/2 and is always relevant in the RG sense. It is associated with a healing (Kondo) length

$$\xi \propto T_B^{-1} \propto (J')^{2/h-2}$$

Unfolding and bosonizing leads to

$$H = \int_{-\infty}^{\infty} dx \sum_{a=1,2} (\partial_x \phi_a)^2 + \lambda \left[S^- \sum_{a=1,2} e^{i\beta\phi_a(0)/\sqrt{2}} + h.c. \right] \qquad h = \frac{\beta^2}{8\pi}, \ \lambda \propto J'$$

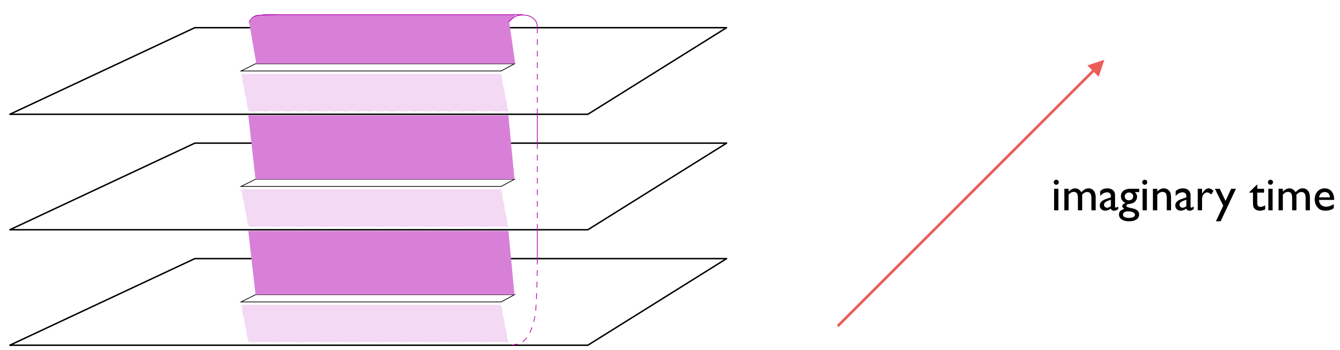
Is S universal along the RG flow? How does it vary as a function of L, l, T_B ?

The tricks of conformal field theory can be extended at least formally to study crossovers: perturbed CFT

Replica trick: $S_A = -\lim_{n\to 1} \frac{\mathrm{d}}{\mathrm{d}n} \mathrm{Tr}_{\mathcal{H}_A}(\rho_A)^n$

Continue analytically from $n \in \mathbb{N}$

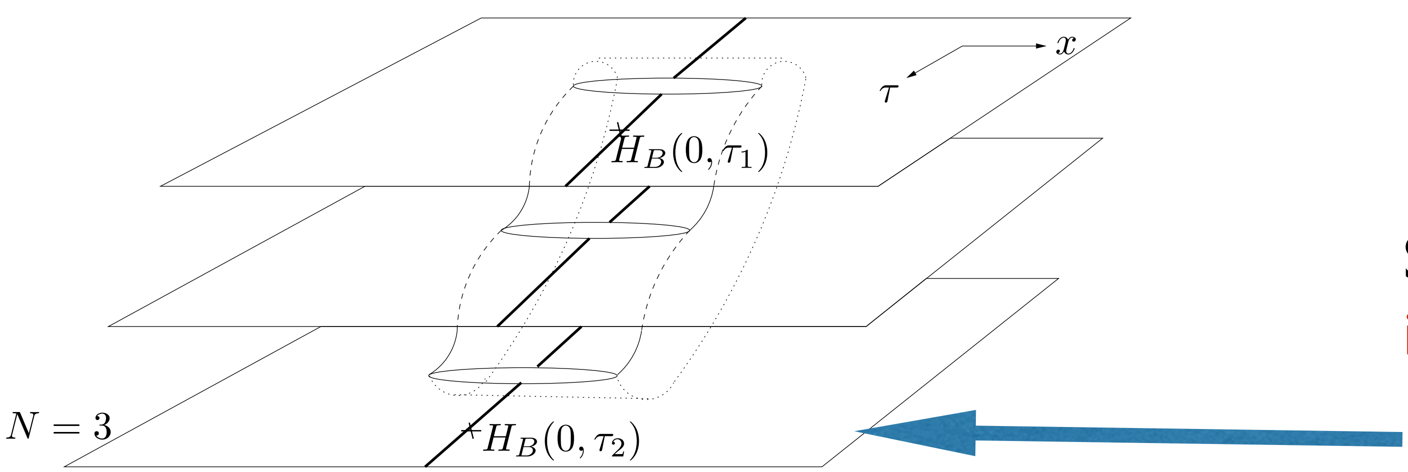
Hence define theory on multi-sheeted Riemann surface



Branch-point twist fields $\mathcal{T}$ at $(x, y) = (a_1, 0)$ and $(a_2, 0)$

$$\mathrm{Tr}_{\mathcal{H}_A}(\rho_A)^n \propto \left\langle \mathcal{T}(a_1, 0)\tilde{\mathcal{T}}(a_2, 0)\right\rangle_{\mathcal{L}^{(n)}}$$

Now need to add perturbation on the Riemann surface



Since it's a T=0 problem, integrals run over infinite imaginary time direction

The Reny entropy is essentially the two point function of twist operators.

As a result, like for most p point functions in PCFT, there are no UV divergences if D= $\frac{h}{2}<\frac{1}{2}$

But there are strong IR divergences: $\int_0^\infty \frac{d\tau}{\tau^{2D}}$ diverges in 0 or ∞

the size L does not act as a cutoff. For entanglement, divergences appear at first non trivial order: the leading correction to S is thus non analytic in J'!

Contrast with boundary (Affleck Ludwig) entropy where finite T act as cutoff

Compare with screening cloud (Affleck Barzykin 1997)

General field theoretic arguments however suggest that, since we consider a two point function

$$\mathrm{Tr}\ (\rho_A)^n = c_n(aT_B)\left(\frac{L}{a}\right)^{-\frac{1}{6}(n-n^{-1})}\Omega(LT_B, lT_B, n)$$

non universal part

and thus

$$\frac{\partial S(\alpha = \ell/L, L)}{\partial \ln L} = f(LT_B, \ell/L)$$

The only universal part
"effective" central charge

Note: $S \neq c_{\text{eff}}\left(LT_B, \frac{l}{L}\right) \times \ln \frac{L}{\epsilon}$

However, in the symmetric (Kondo like) case, scaling is better

$$S(L/2, L) - S_{IR} = g(LT_B)$$

No known way to calculate or approximate f or g analytically except in integrable cases

Entanglement in integrable QFT

- Integrability means here existence of a basis of excitations whose dynamics is factorized on two body processes, with no particle production either in the bulk or in the interaction with the impurity. While most often used in the massive case, this basis exists also in the massless case.

- The excitations (quasiparticles) are L or R moving, and live on the n copies of the initial theory

- The two point function of the twist operators is calculated by expanding on the qp basis and requires knowledge of form-factors (Cardy, Castro-Alvaredo, Doyon 2007)

$$\mathrm{Tr}_{\mathcal{H}_A}(\rho_A)^n \propto \left\langle \mathcal{T}(a_1,0)\tilde{\mathcal{T}}(a_2,0)\right\rangle_{\mathcal{L}^{(n)}}$$

insert sum over intermediate states

$$\sum_{k=0}^{\infty}\int_{-\infty}^{\infty}\prod_{i=1}^{k}d\theta_i\sum_{\epsilon_i=1}^{N}|\theta_1\ldots\theta_k>_{\epsilon_i}\ {}_{\epsilon_i}<\theta_k\ldots\theta_1|$$

replica index

The expansion in terms of k converges very fast, for all values of L and impurity couplings

(even though the theory is massless)

A simple example: the RLM (XX chains + dot). Still non trivial since twist fields are very non local in terms of the (n copies of) fermions.

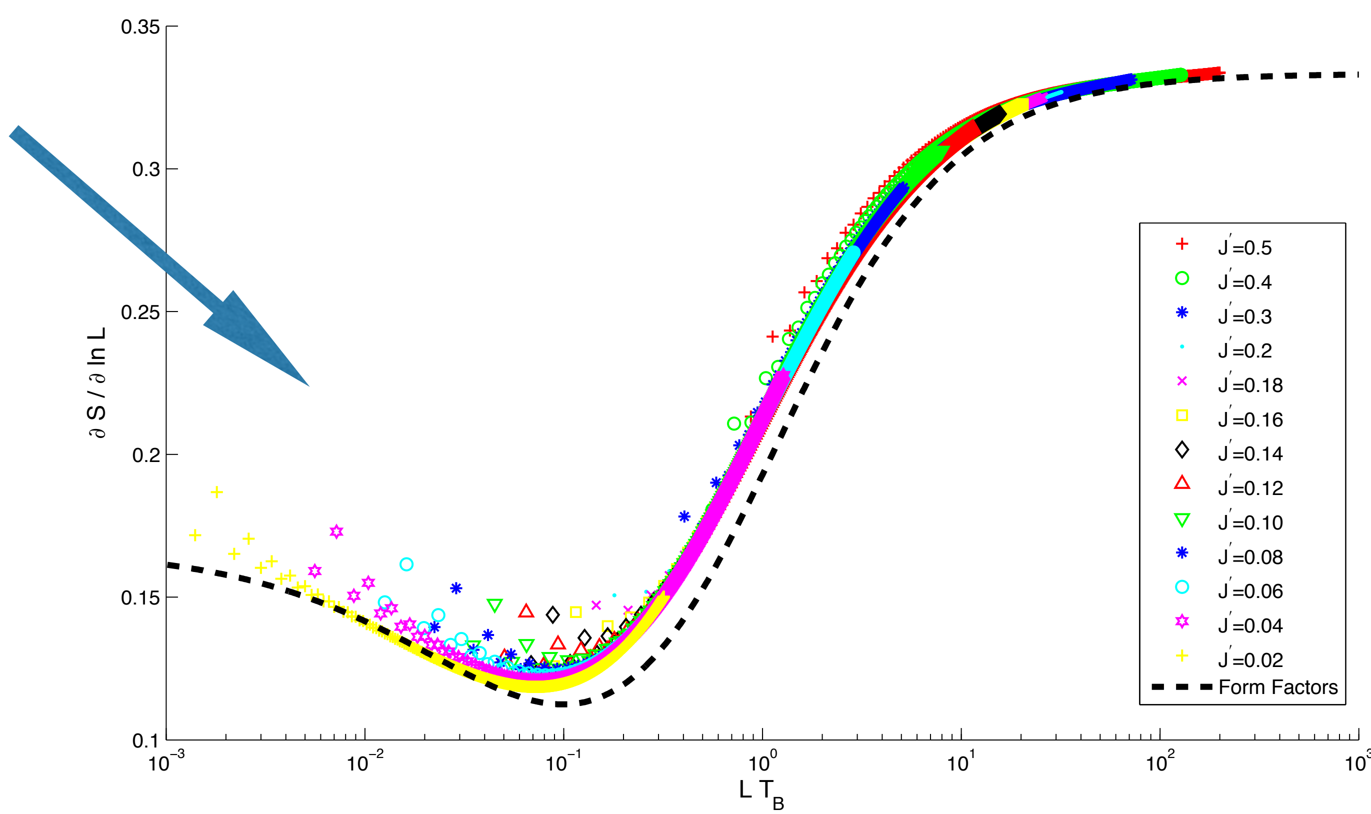
$$\frac{\partial S_A(\alpha)}{\partial \log L} = F(LT_B)$$

where

$$F(x) = \frac{2}{3}\int_0^\infty \mathrm{d}v\, \mathrm{e}^{-2v}\left(\frac{x}{v+x}\right)^2 + \frac{2}{3}\int_0^\infty \mathrm{d}v\, \left(\alpha \mathrm{e}^{-4\alpha v} + (1-\alpha)\mathrm{e}^{-4(1-\alpha)v}\right)\left(\frac{v}{v+x}\right)^2 + \ \ldots$$

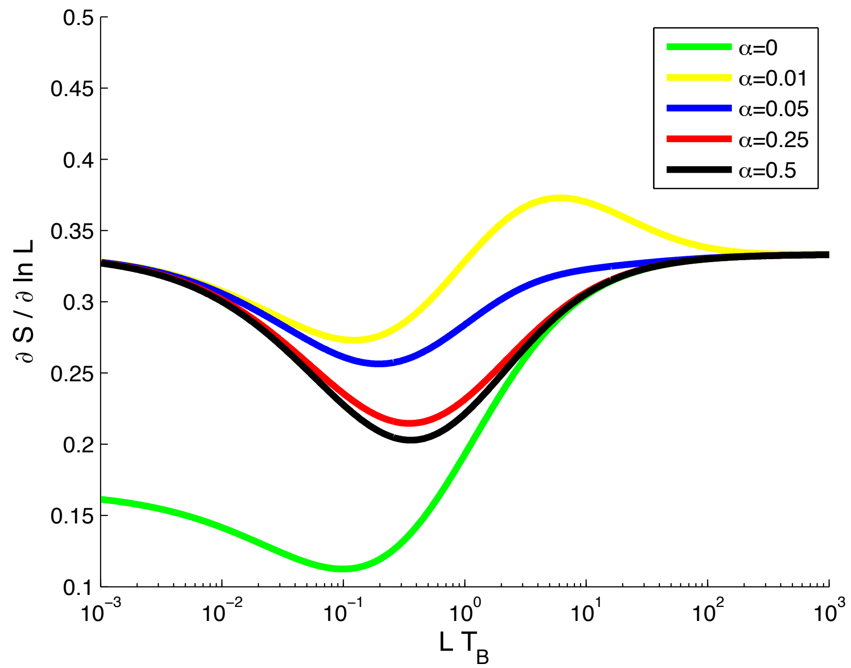
Numerics and FF for $\alpha = 0$

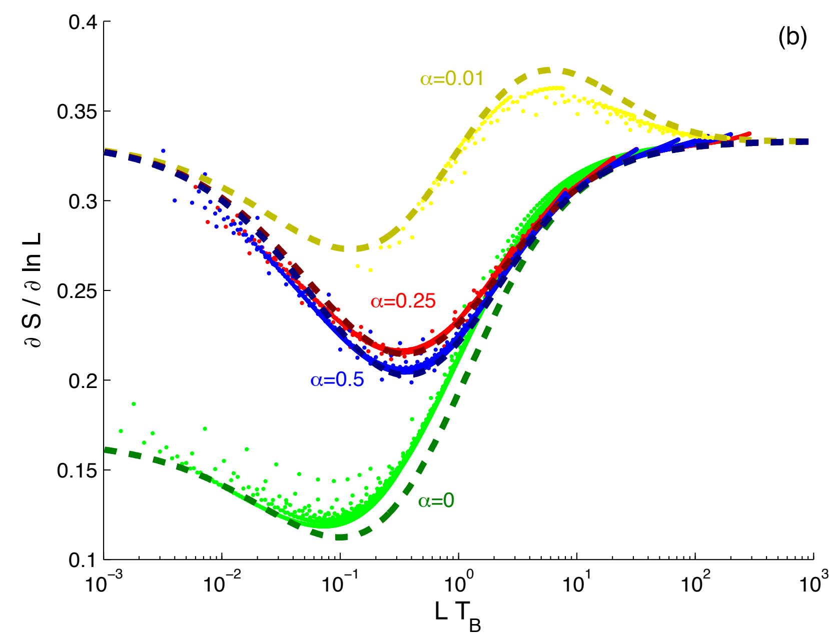
Effective central charge is non monotonic !



Chains of 32000 sites.
No adjustable parameter

Changing α





Discontinuous behavior for α=0

UV limit ($x \ll 1$):

$$F(x) = \frac{1}{6} + \frac{4}{3}\left(1 + \gamma + \log(4x)\right) x + \mathcal{O}(x^2)$$

The non perturbative dependence

$$x = LT_B,\ T_B \propto (J')^2$$

Integrability can also mean full knowledge of the low energy action, and a meaningful ‘’irrelevant’’ perturbation theory (Lesage Saleur 1996). Eg for anisotropic Kondo (the previous case with $\alpha=1/2$ then describes the Toulouse point)

$$S(L/2,L)-S_{IR}=g(LT_B) \quad = \quad \frac{1}{6}\ln\left(1+\frac{1}{LT_{\mathrm{B}}}\right)-\frac{18}{35}\frac{(\pi g_4)^2}{(2LT_{\mathrm{B}})^6}\left(1-\frac{6}{LT_{\mathrm{B}}}\right)(4\alpha^4-8\alpha^2+9)+\mathcal{O}((LT_{\mathrm{B}})^{-8})$$

Leading order: Affleck Laflorencie Sorensen 2006

$$g_4=\frac{D}{6\pi^2}\left(\frac{\Gamma(D/2(1-D))}{\Gamma(1/2(1-D))}\right)^3\frac{\Gamma(3/2(1-D))}{\Gamma(3D/2(1-D))}, \quad \alpha=\frac{(1-D)}{\sqrt{D}}.$$

D the dimension of the perturbation (D= 1 for Kondo, D=1/2 at Toulouse point)

Some results for D=1/2

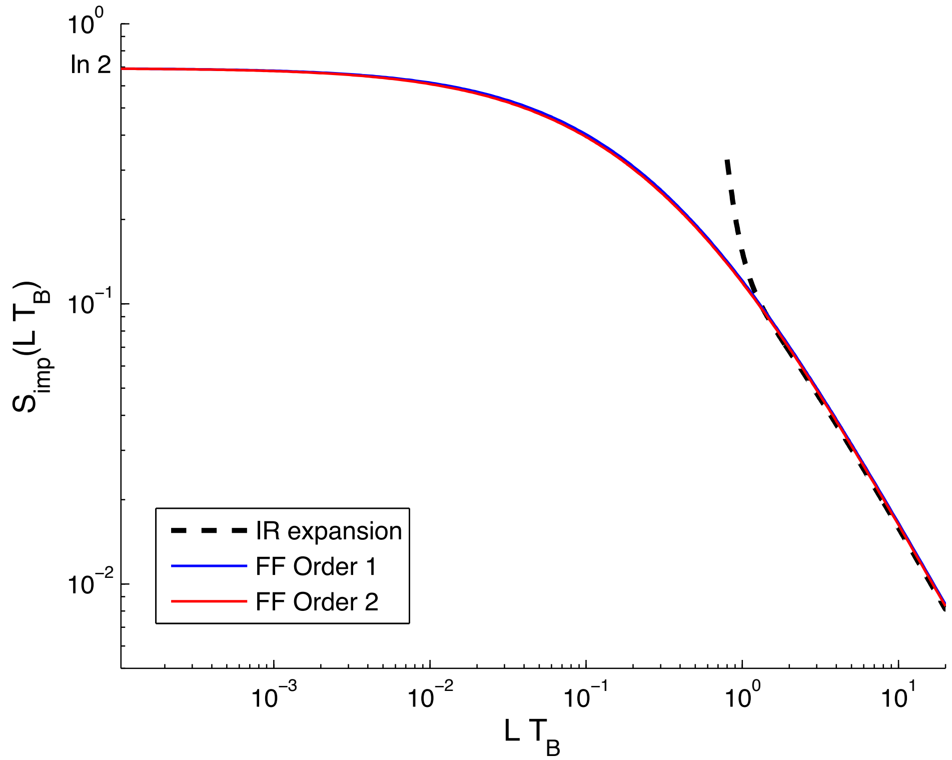
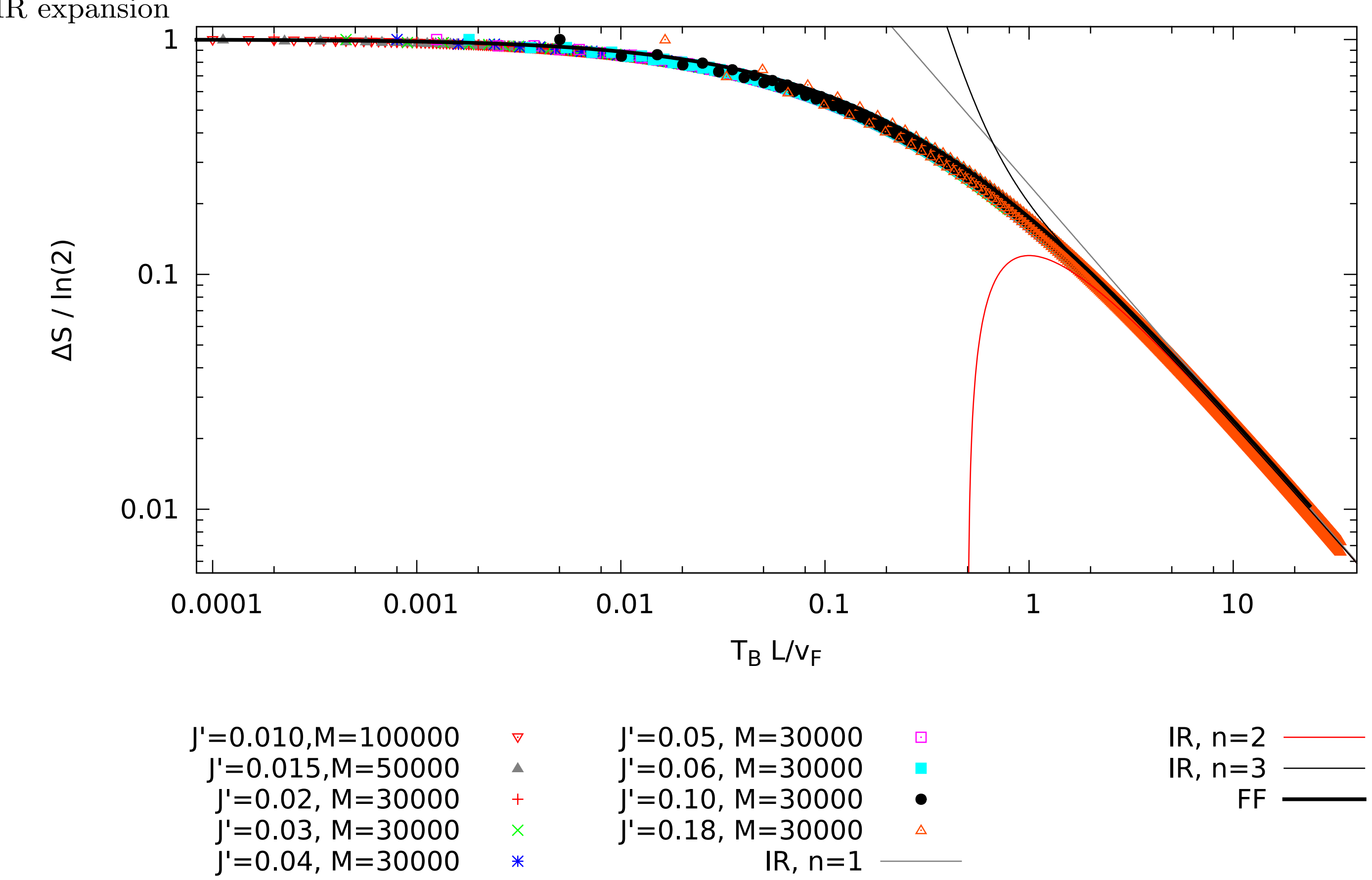


FIG. 1: The form factor approximations together with the IR expansion

Numerics



Some conclusions

- It is possible to understand non analyticity of S as a function of the couplings using CFT arguments

- Apart from integrable cases, there is no known way to get analytical results (not even perturbative) about S.

- General, instanton like expansions? AdS/CFT duality? (Albash, Johnson, Saleur)