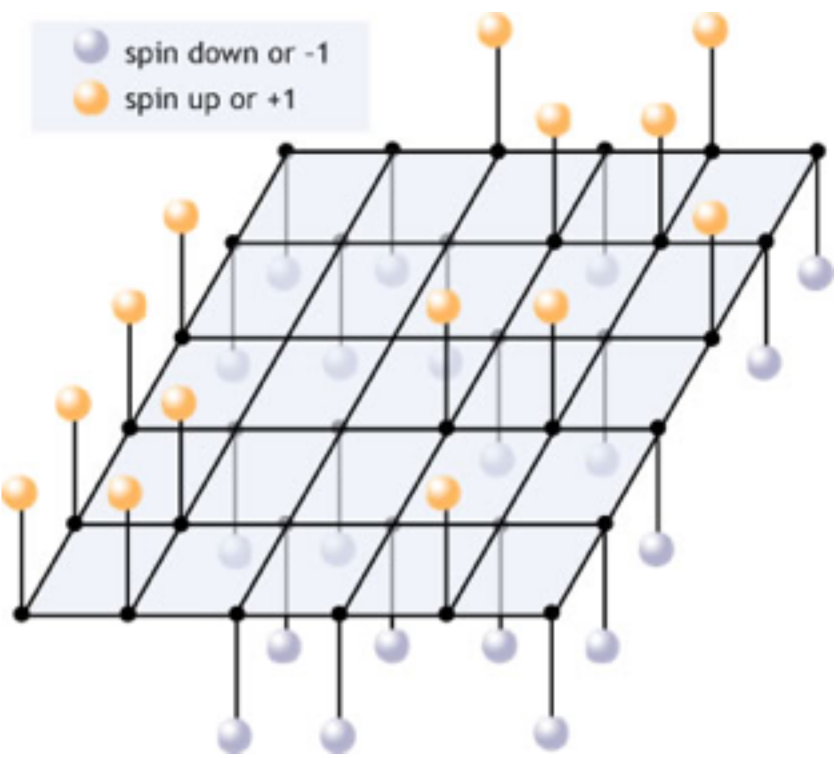
Fusion in the affine Temperley-Lieb algebra

The mandatory logos:

Some physics motivations

(Paul Martin)

- 2 dim Critical statistical mechanics systems like the Ising model



energy $$E(\{S_i\}) = -J \sum_{<i,j>} S_i S_j, \quad S_i = \pm 1$$

probability of a configuration $$P(\{S_i\}) = \frac{e^{-E(\{S_i\}/k_B T}}{Z}$$

partition function $$Z = \sum_{\{S_i\}} e^{-E(\{S_i\}/k_B T}$$

- have critical points where

correlation functions of local observables decay as power laws $$\langle S_{i_1} S_{i_2} \rangle \sim \frac{1}{|r_{i_1} - r_{i_2}|^{2x}}$$ ← scaling dimension properties become universal and can be described by

conformal field theory

(which means in particular that correlation functions have nice properties under conformal transformations)

- **In practice - and in physics - CFT relies on two technical features**

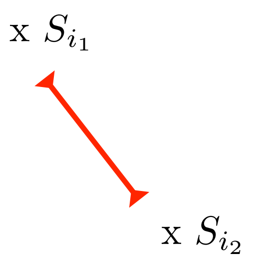
  - Representation theory of the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m}$$

(infinite dimensional Lie algebra with central extension)

  - the conformal bootstrap

qualitative idea: the product of two local observables seen at large distance should behave like a sum of local observables (with well defined scaling dimensions) - the Operator Product Expansion (OPE) example:



$\sim \quad \epsilon \equiv$ energy $(= S_{i_1} S_{i_2},\ i_1, i_2$ neighbors$)$

this gets formalized in the language of vertex operator algebras (VOA)

- **There are reasons why it is important to understand better these two aspects directly on the lattice - where, for instance, the conformal symmetry cannot be exact.**

one of these reasons is that we don't understand
much to Logarithmic CFT (non semi-simple cases)

In quantum field theory, unitarity is mandatory. It implies semi-simplicity, and, in many cases, allows full classification of Virasoro modules that can appear (e.g. c<1 classification, Friedan Qiu Shenker, Rocha Caridi, Feigin Fuchs)

In statistical mechanics, there is no such constraint. Percolation, Self-avoiding walks,disordered electronic systems all correspond to non-unitary CFTs. This translates into non semi-simple Virasoro representation theory. And Virasoro is wild (Germoni).

The hope is that we can understand what kind of algebraic properties to expect in the CFT from those we can investigate analytically/numerically on the lattice. That's the "associative algebraic approach to LCFT" (Read Saleur 2001)

- **Now, the parallels between the Virasoro and the Temperley-Lieb algebra are plenty (see later)**

in particular, Temperley-Lieb appears in the detailed description of the lattice models (eg, in the construction of the transfer matrix/Hamiltonian), and generators are roughly like the stress-energy tensor

raising the question

- **what is the TL analog of OPEs?**

Fusion in open TL

- In CFT, there are really two Virasoro algebras $L_n, \ \bar{L}_n$. That's because physical fields $\Phi(z,\bar{z})$ are non chiral.

- There is however a situation where physical fields are chiral, the so called Boundary CFT (BCFT).

this should correspond to the ordinary TL algebra

The (finite) Temperley–Lieb (TL) algebra $\mathsf{TL}_N(m)$ is an associative algebra over $\mathbb{C}$ generated by unit $\mathbf{1}$ and e_j, with $1 \leq j \leq N-1$, satisfying the defining relations

$$\begin{aligned} e_j^2 &= m e_j, \\ e_j e_{j\pm1} e_j &= e_j, \qquad\qquad (2.1) \\ e_j e_k &= e_k e_j \qquad (j \neq k,\ k \pm 1). \end{aligned}$$

This algebra has a well-known faithful diagrammatical representation in terms of non-crossing pairings on a rectangle with N points on each of the opposite sides. Multiplication is performed by placing two rectangles on top of each other, and replacing any closed loops by a factor m. While the identity corresponds to the diagram in which each point is directly connected to the point above it, the generator e_i is represented by the diagram, see Fig. 2, where the points i on both sides of the rectangle are connected to the point $i+1$ on the same side, all other points being connected like in the identity diagram. The defining relations are easily checked by using isotopy ambient on the boundary of the rectangle, see Fig. 3.

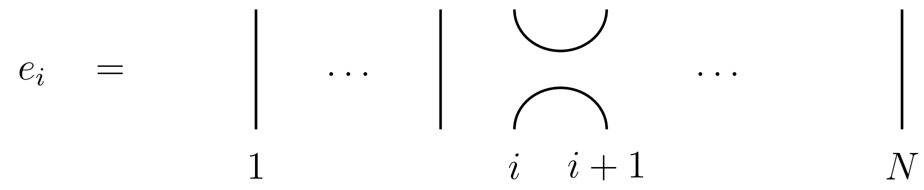


FIGURE 2. The diagrammatic representation of e_i.

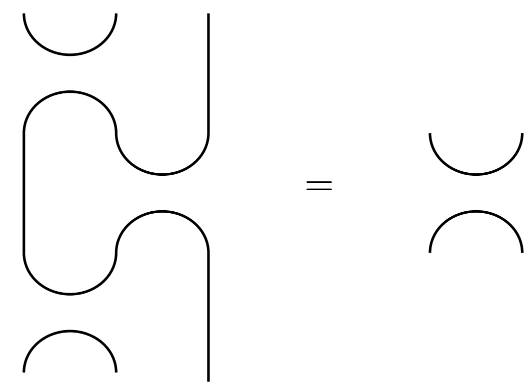
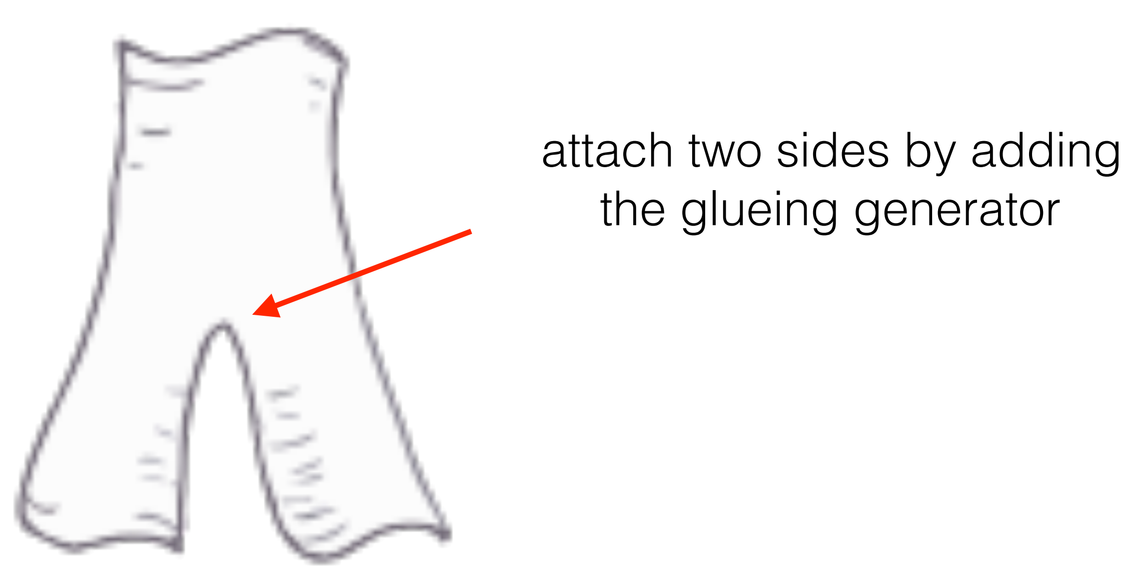


FIGURE 3. The diagrammatic version of the relation $e_i e_{i+1} e_i = e_i$.

- Fusion in this case was defined in Read Saleur 2001



Definition 2.2.1 ([21, 29]). *Let M_1 and M_2 be two modules over TL_{N_1} and TL_{N_2} respectively. Then, the tensor product $M_1 \otimes M_2$ is a module over the product $\mathsf{TL}_{N_1} \otimes \mathsf{TL}_{N_2}$ of the two algebras. Using the standard embedding, we consider this product of algebras as a subalgebra in TL_N, for $N = N_1 + N_2$. The fusion (bi-)functor*

$$\times_f : \quad \mathsf{C}_{N_1} \times \mathsf{C}_{N_2} \to \mathsf{C}_{N_1+N_2} \tag{2.4}$$

on two modules M_1 and M_2 is then defined as the module induced from this subalgebra, i.e.,

$$M_1 \times_f M_2 = \mathsf{TL}_N \otimes_{\left(\mathsf{TL}_{N_1} \otimes \mathsf{TL}_{N_2}\right)} M_1 \otimes M_2 \ , \tag{2.5}$$

where we used the balanced tensor product over $\mathsf{TL}_{N_1} \otimes \mathsf{TL}_{N_2}$.

Straightforward results in the generic case.

2.3. **Standard and projective TL modules.** We also recall *the standard* $\mathsf{TL}_N(m)$ modules $\mathcal{W}_j[N]$ of the weight $x \leq j \leq N/2$, where $x = \frac{1}{2}(N \bmod 2)$. First, we need to introduce "half-diagrams" (usually called link states) obtained from Temperley-Lieb diagrams (i.e., non-crossing pairings on a rectangle with N points on each of the opposite sides) and cutting these diagrams horizontally in the middle. Each half has N points: some of them are connected by arcs, and some others are not connected to anything. The latter are often called *through-lines* (or defects). The algebra acts in the obvious diagrammatic way by concatenating Temperley-Lieb diagrams with link states, eliminating all loops in price of multiplying the diagram by m^n, where n is the number of loops, and keeping track of the connectivities using isotopy. It is clear that the number of through-lines cannot increase under the action of the algebra. Standard modules $\mathcal{W}_j[N]$ are obtained by letting the algebra act as usual when the number of through-lines – denoted by $2j$ – is conserved, and setting this action to zero when the number of through-lines decreases. It is well known that these modules are irreducible for $\mathfrak{q}$ generic, while their dimension is given by differences of binomial coefficients

$$d_j[N] = \binom{N}{\frac{N}{2}+j} - \binom{N}{\frac{N}{2}+j+1}. \tag{2.7}$$

$$\mathcal{W}_{j_1}[N_1] \times_f \mathcal{W}_{j_2}[N_2] = \bigoplus_{|j_1-j_2|}^{j_1+j_2} \mathcal{W}_j[N_1+N_2]$$

SU(2)q, Schur-Weyl

- **Complex and fascinating results for $\mathfrak{q}$ a root of unity $(m = \mathfrak{q} + \mathfrak{q}^{-1})$**

of particular interest in the physics literature has been fusion of projective modules

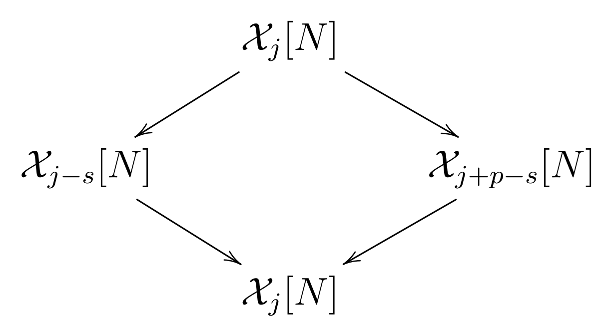
(they seem to be what matters for physics)

$\mathfrak{q} = e^{i\pi/p}$

$$\mathcal{W}_j[N] = \mathcal{X}_j[N] \longrightarrow \mathcal{X}_{j+p-s}[N]$$

$s \equiv s(j) = (2j+1) \bmod p$

$\mathcal{P}_j[N] \quad =$



Kytola, Ridout, St Aubin, Kausch, Gaberdiel, Nahm, Pearce, Rasmussen, Belletete, Jacobsen, Gaynutdinov, Read, Saleur [2007-2016]

“matches” fusion of staggered Virasoro modules in LCFT

precise categorical equivalence Gaynutdinov Saleur 2016

Fusion in affine TL

Gaynutdinov, Jacobsen, Saleur, 2016

- We now go back to the bulk (non-boundary) case. This should correspond to a TL algebra acting on a periodic system, the affine TL (Martin-Saleur 93, Jones 94, Green 98, Erdmann Green 99)

3.1.1. *Definition I: generators and relations. The affine Temperley–Lieb (aTL) algebra* $\mathsf{T}^a_N(m)$ is an associative algebra over $\mathbb{C}$ generated by u, u^{-1}, and e_j, with $j \in \mathbb{Z}/N\mathbb{Z}$, satisfying the defining relations

$$\begin{aligned} e_j^2 &= m e_j, \\ e_j e_{j\pm1} e_j &= e_j, \\ e_j e_k &= e_k e_j \qquad (j \neq k,\ k \pm 1), \end{aligned} \tag{3.1}$$

which are the standard TL relations but defined for the indices modulo N, and

$$\begin{aligned} u e_j u^{-1} &= e_{j+1}, \\ u^2 e_{N-1} &= e_1 \dots e_{N-1}, \end{aligned} \tag{3.2}$$

where the indices $j = 1, \dots, N$ are again interpreted modulo N.

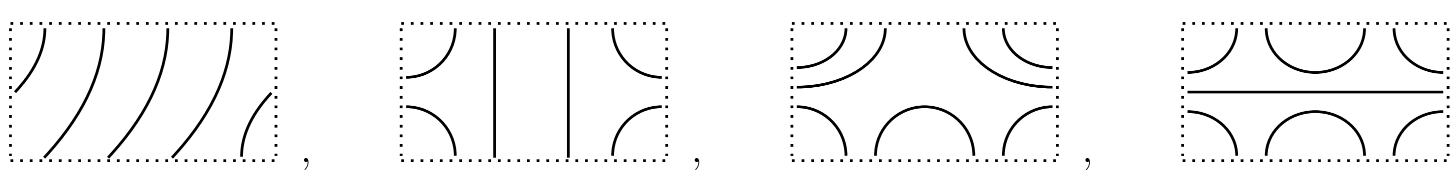
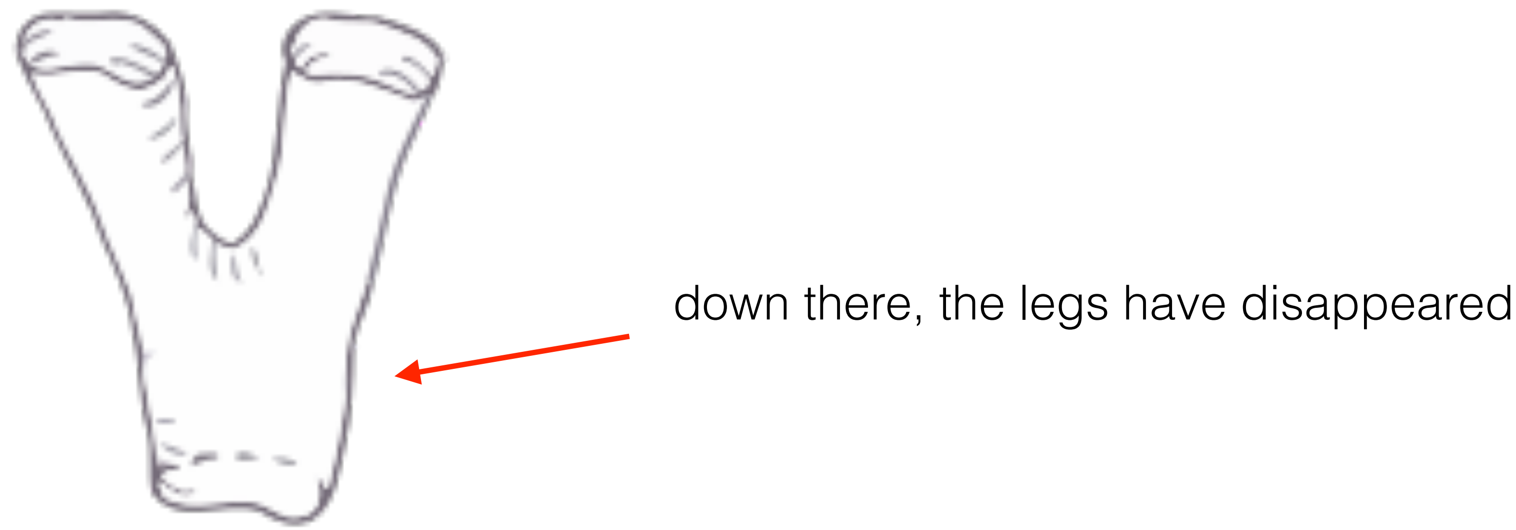


FIGURE 4. Examples of affine diagrams for $N = 4$, with the left and right sides of the framing rectangle identified. The first diagram represents the translation generator u while the second diagram is for the generator $e_4 \in \mathsf{T}^a_4(m)$. The third and fourth ones are examples of $j = 0$ diagrams.

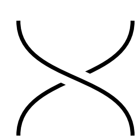
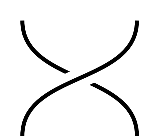
- Note that diagrams in this algebra allow winding of through lines around the annulus any number of times, and different windings result in independant algebra elements. Moreover, in the ideal of zero through lines, any number of non-contractible loops is allowed. The algebra is thus infinite dimensional.

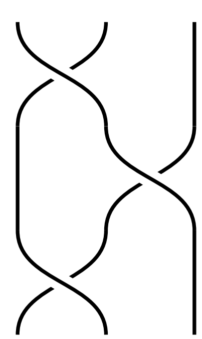
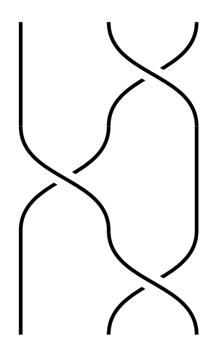
- Fusion in this affine case requires glueing two cylinders. How do we do this without cutting them open?



- Affine braid group

$$g_i^{\pm 1} = \pm \mathrm{i}(\mathfrak{q}^{\pm\frac{1}{2}}\mathbf{1} - \mathfrak{q}^{\mp\frac{1}{2}} e_i)$$

g_i =  g_i^{-1} = 

 =  $(g_i g_{i\pm1} g_i = g_{i\pm1} g_i g_{i\pm1})$

The trick   Gaynutdinov Saleur 2016

we can embed the product of two affine TL algebras, $\mathsf{T}^a_{N_1}$ and $\mathsf{T}^a_{N_2}$, into T^a_N with $N = N_1 + N_2$. Let us denote the generators in the ith algebra as $u^{(i)}$ and $e^{(i)}_j$, with $i = 1, 2$, and use standard notations for the generators in the "big" algebra T^a_N. We first define the map on the TL generators $e^{(i)}_j$, where $j \neq 0$, in the standard way

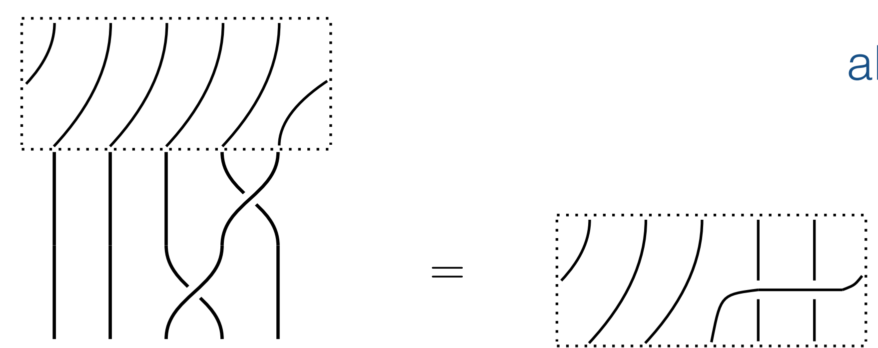
$$e^{(1)}_j \mapsto e_j, \qquad e^{(2)}_k \mapsto e_{N_1+k}, \qquad 1 \le j \le N_1 - 1, \quad 1 \le k \le N_2 - 1. \tag{3.7}$$

The translation generators $u^{(1)}$ and $u^{(2)}$ are mapped as (recall, we set $N = N_1 + N_2$)

$$u^{(1)} \mapsto u\, g^{-1}_{N-1} \dots g^{-1}_{N_1}, \qquad u^{(2)} \mapsto g_{N_1} \dots g_1 u. \tag{3.8}$$

in terms of diagrams:

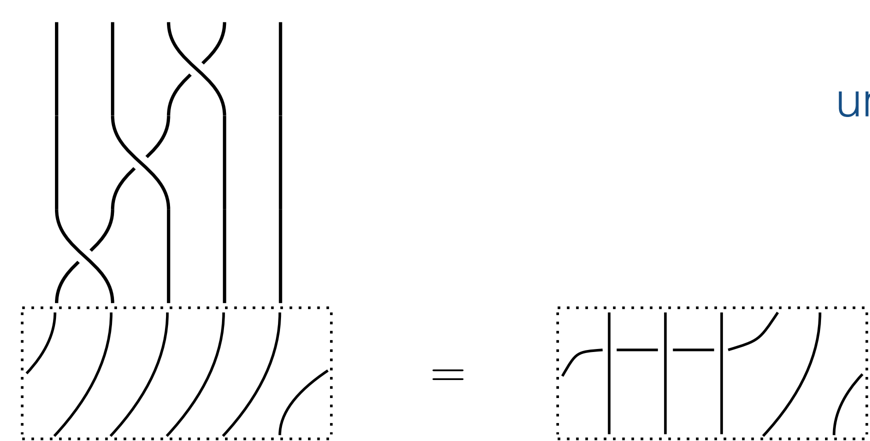
(3.9) $u^{(1)} \mapsto$



above

where we assumed that $N_1 = 3$ and $N_2 = 2$, and for the second translation $u^{(2)}$ we have the diagram

(3.10) $u^{(2)} \mapsto$



under

- One can check that:

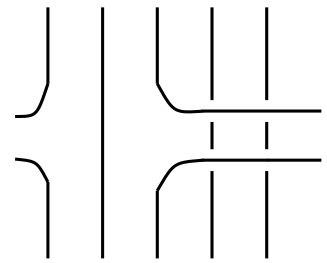
$$u^{(1)}u^{(2)} = u^{(2)}u^{(1)}$$

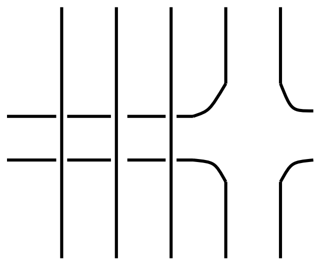
$$\left(u^{(1)}\right)^2 e_{N_1-1} = e_1 \dots e_{N_1-1}, \qquad \left(u^{(2)}\right)^2 e_{L-1} = e_{N_1+1} \dots e_{L-1}$$

- Next, we define the map on the periodic TL generators

$$\begin{aligned} e_0^{(1)} &\mapsto g_{N_1} \dots g_{N-1}\, e_0\, g_{N-1}^{-1} \dots g_{N_1}^{-1}, \\ e_0^{(2)} &\mapsto g_0^{-1} \dots g_{N_1-1}^{-1}\, e_{N_1}\, g_{N_1-1} \dots g_0 \end{aligned}$$

in terms of diagrams:

$e_0^{(1)} =$ 

$e_0^{(2)} =$ 

- One can check that:

$$\left(e_0^{(i)}\right)^2 = (\mathfrak{q} + \mathfrak{q}^{-1}) e_0^{(i)}, \qquad i = 1, 2.$$

$$e_0^{(i)} = u^{(i)} e_{N_i-1}^{(i)} \left(u^{(i)}\right)^{-1} = \left(u^{(i)}\right)^{-1} e_1^{(i)} u^{(i)}, \qquad i = 1, 2,$$

$$e_0^{(i)} e_1^{(i)} e_0^{(i)} = e_0^{(i)}, \qquad e_1^{(i)} e_0^{(i)} e_1^{(i)} = e_1^{(i)},$$
$$e_0^{(i)} e_{N_i-1}^{(i)} e_0^{(i)} = e_0^{(i)}, \qquad e_{N_i-1}^{(i)} e_0^{(i)} e_{N_i-1}^{(i)} = e_{N_i-1}^{(i)},$$

$$e_0^{(1)} e_0^{(2)} = e_0^{(2)} e_0^{(1)}$$

this holds only because of the above/under pattern

So now we can do fusion:

Definition 4.1. *Let M_1 and M_2 be two modules over $\mathsf{T}^a_{N_1}(m)$ and $\mathsf{T}^a_{N_2}(m)$ respectively. Then, the tensor product $M_1 \otimes M_2$ is a module over the product $\mathsf{T}^a_{N_1}(m) \otimes \mathsf{T}^a_{N_2}(m)$ of the two algebras. Using the embedding (3.24), we consider this product of algebras as a subalgebra in $\mathsf{T}^a_N(m)$, for $N = N_1 + N_2$. The (affine) fusion functor $\widehat{\times}_f$ on two modules M_1 and M_2 is then defined as the module induced from this subalgebra, i.e.*

$$M_1 \,\widehat{\times}_f\, M_2 = \mathsf{T}^a_N \otimes_{\left(\mathsf{T}^a_{N_1} \otimes \mathsf{T}^a_{N_2}\right)} M_1 \otimes M_2, \tag{4.1}$$

where we used the balanced tensor product over $\mathsf{T}^a_{N_1} \otimes \mathsf{T}^a_{N_2}$ and we abuse the notation by writing T^a_N instead of $\mathsf{T}^a_N(m)$.

The results are a bit complicated. First, introduce standard modules $\mathcal{W}_{j,z}[N]$ (Martin-Saleur/ Graham Lehrer)

Here 2j is the number of through lines as usual. z is a complex number whose role is to `unwind' through lines that go around the cylinder: whenever the 2j lines go arond clockwise we unwind them at the price of a factor 1/z;counterclockwise leads to a factor z instead. Finally, for j=0, non contractible loops are eliminated for a factor z+1/z

$$\hat{d}_j[N] \equiv \dim \mathcal{W}_{j,z}[N] = \binom{N}{\frac{N}{2} + j}$$

we then have the conjectured results (based on direct calculations)

$$\mathcal{W}_{j_1,z_1}[N_1] \,\widehat{\times}_f\, \mathcal{W}_{j_2,z_2}[N_2] = \mathcal{W}_{j,z}[N_1+N_2]$$

no sum!

- For $j = j_1 + j_2$ and any values of j_1, j_2:

$$z_1 = (i\sqrt{\mathfrak{q}})^{-2j_2} z^{+1}\,, \qquad z_2 = (i\sqrt{\mathfrak{q}})^{+2j_1} z^{+1}$$

- For $j = j_1 - j_2$ and either $j = 0$ or $j_2 > 0$:

$$z_1 = (i\sqrt{\mathfrak{q}})^{+2j_2} z^{+1}\,, \qquad z_2 = (i\sqrt{\mathfrak{q}})^{-2j_1} z^{-1}$$

- For $j = j_2 - j_1$ and either $j = 0$ or $j_1 > 0$:

$$z_1 = (i\sqrt{\mathfrak{q}})^{+2j_2} z^{-1}\,, \qquad z_2 = (i\sqrt{\mathfrak{q}})^{-2j_1} z^{+1}$$

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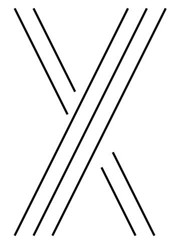
otherwise fusion is zero

- this fusion is non-commutative, and associative

- this exists another fusion $\widehat{\times}_f^-$ obtained by switching over and under, and the two are related by braiding



:

$$M_1[N_1] \,\widehat{\times}_f\, M_2[N_2] \xrightarrow{\;\cong\;} M_2[N_2] \,\widehat{\times}_f^-\, M_1[N_1]$$

- a technical remark: it is well known how affine TL can be obtained as a quotient of affine Hecke. There is meanwhile a well known fusion in affine Hecke, Zelevinsky tensor product. The problem is, that this tensor product and the quotient to get affine TL are not, in general, compatible (so the result is "zero"). We have checked that, when it is compatible, our results are recovered.

there's room for a theorem!

Affine TL fusion in the conformal limit

- It is possible to define the "scaling limit" of the ATL modules. This is done by considering a lattice model whose critical Boltzmann weights provide, in the transfer matrix description, a representation of ATL (see Paul Martin's book!). In a nutshell, we take a Hamiltonian

$$H = -\sum_i e_i$$

while the logarithm of the translation generator gives us the momentum P. The generating function of their spectra gives us characters of Virasoro

$$\mathrm{Tr}\ e^{-\beta_R(H-Ne_0)}e^{-i\beta_I P} \xrightarrow{N\to\infty} \mathrm{Tr}\, q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}$$

trace taken over modules of ATL

trace taken over modules of $\mathrm{Vir}\times\overline{\mathrm{Vir}}$

where $q(\bar{q}) = \exp\left[-\frac{2\pi}{N}(\beta_R \pm i\beta_I)\right]$ (Cardy)

- One finds, then that our fusion corresponds to glueing the right component of one field with the left component of the other field. Schematically:

Verma modules with Virasoro highest weight h,h'

$$\mathcal{W}_{j,z} \mapsto \mathrm{Vir}_h \times \overline{\mathrm{Vir}}_{h'}$$

where, setting $\mathfrak{q} = e^{\frac{i\pi}{x+1}}$, $c = 1 - \frac{6}{x(x+1)}$ and h,h' are functions of j,z

$$\left(\mathrm{Vir}_h \times \overline{\mathrm{Vir}}_{h'}\right) \,\widehat{\times}_f\, \left(\mathrm{Vir}_{h'} \times \overline{\mathrm{Vir}}_{h''}\right) = \left(\mathrm{Vir}_h \times \overline{\mathrm{Vir}}_{h''}\right)$$

the same conformal weight

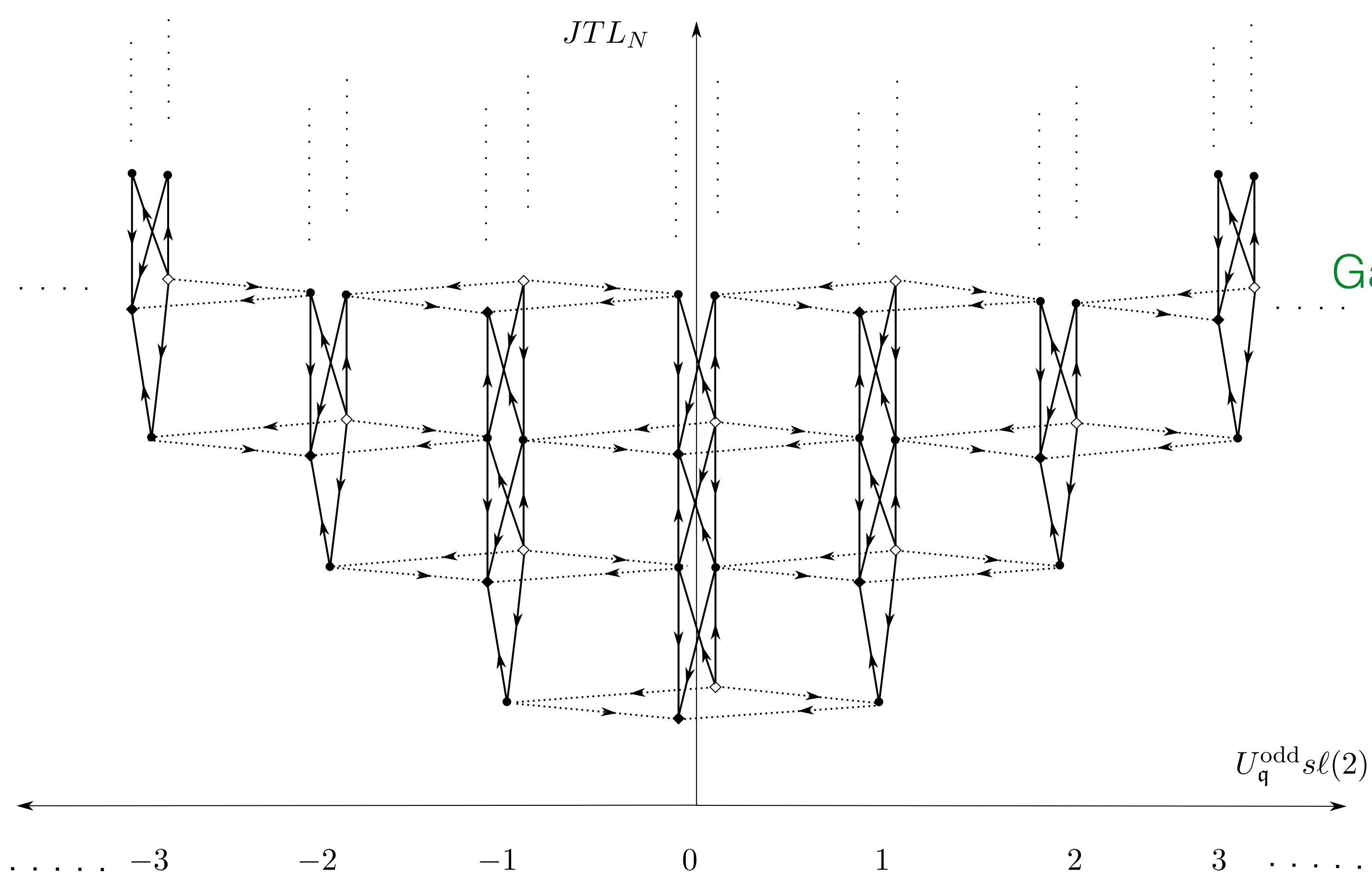
- We don't yet understand what to do with this. It may be our fusion is not the right one for physical applications.

Conclusions

- Physicists have to do representation theory to understand in detail the relationship between lattice models and their conformal invariant limits. This is particularly crucial to make progress on logarithmic CFTs (non semi-simple VOAs) which play a role in the description of many systems of interest (in particular those involving disorder)

- Apart from modules and fusion, another hot topic is the understanding of lattice models "Hilbert spaces" as bimodules over ATL and its centralizer

(m=0,gl(1/1))



Gaynutdinov, Read, Saleur 2013

- Questions for mathematicians: Fusion in other algebras? Blob/boundary Temperley-Lieb/Temperley-Lieb type B,C

- ATL and affine quantum groups?