Quantum entanglement in non-hermitian critical spin chains and non-unitary CFTs

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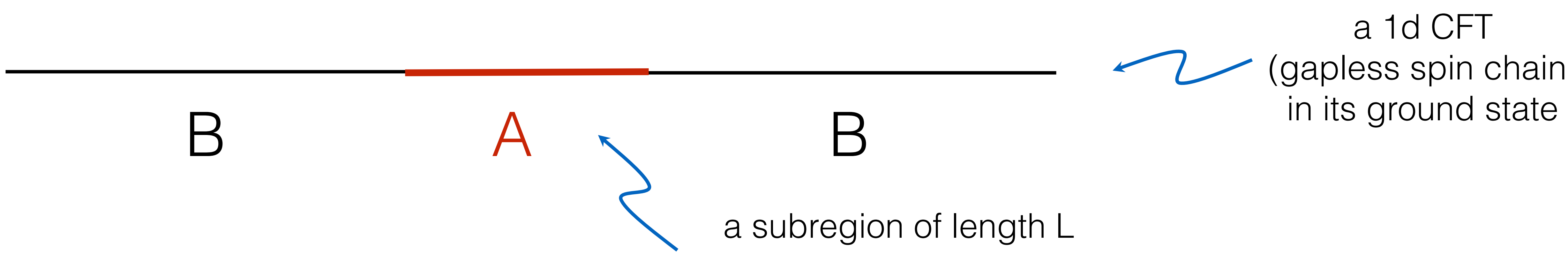
Entanglement and CFT

- Concepts of quantum information are profoundly affecting our understanding of and approach to quantum many-body systems
- This is especially true of quantum critical points in 1d where these concepts can be merged with CFT ideas

One of the main concepts in this case is entanglement entropy



In the ground state we have $\rho = |0\rangle\langle 0|$. Define the reduced density operator $\rho_A = \mathrm{Tr}_B \rho$

and the (Von Neumann entanglement entropy EE) $S_A = -\mathrm{Tr}_A \rho_A \ln \rho_A$

The key result is $\boxed{S_A \approx \frac{c}{3} \ln \frac{L}{a}}$ with c the central charge

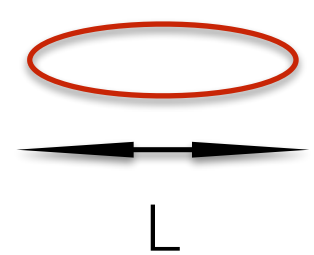
- Non-hermitian “Hamiltonians” are (increasingly) important:

  - phenomenological description of open quantum systems (imaginary terms on the boundary)
  - many 2d statistical mechanics systems in their 1+1d formulation such as SAW, percolation, hard hexagons (Yang-Lee singularity)
  - some 2+1d quantum mechanical systems in their 1+1d formulation such as plateau transitions in several classes of top. insulators
  - all the field theories and spin chains on supermanifolds

What to do with entanglement in this case?

- A reminder from non-unitary CFT: the central charge also appears in the scaling of the ground state energy on a circle

$$E_0 = -\frac{\pi c}{6L}$$



- In minimal non-unitary CFTs $c = 1 - 6\frac{(p-p')^2}{pp'}$ $|p-p'|>1$

there is a state of negative conformal weight $h_{\min} = \frac{1-(p-p')^2}{4pp'}$

and the "real ground state" scales with the "effective central charge" $c_{\text{eff}} = c - 24h_{\min}$

(for example Yang-Lee universality class is $p=5, p'=2, \quad c = -\frac{22}{5}, \quad c_{\text{eff}} = \frac{2}{5}$) [Itzykson,Saleur,Zuber 1986]

- It is natural to expect that, in these cases at least, we will have $\boxed{S_A \approx \frac{c_{\text{eff}}}{3}\ln(L/a)}$

- But it is far from obvious. In particular, recall that the entanglement does not depend on the (conformal state) in which it is calculated! [Alcaraz,Berganza,Sierra 2011]

  a more sophisticated explanation (based on modified twist fields in the replica approach is proposed in [Doyon, Castro-Alvaredo, Ravanini, Bianchini, Levi 2014]

but the issue is full of surprises

Entanglement in quantum group symmetric chains

- We'll consider chains $(\mathbb{C}^{\otimes 2})^{\otimes N}$ and "hamiltonians" with $U_q sl(2)$ symmetry ($q \in \mathbb{C},\ |q| = 1$) which can be expressed in terms of the Temperley-Lieb generators

$$e_i = -\frac{1}{2}[\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \frac{q+q^{-1}}{2}(\sigma_i^z \sigma_{i+1}^z - 1) + \frac{q-q^{-1}}{2}(\sigma_i^z - \sigma_{i+1}^z)]$$

Some simple exercises.

Take $N = 2$ and $H = -e_1$. Although H is not hermitian its eigenvalues are real $E^{(0)} = -(q+q^{-1})$ and $E^{(1)} = 0$

- The ground state ($H|0\rangle = E^{(0)}|0\rangle$) is $|0\rangle = \frac{1}{\sqrt{2}}(q^{-1/2}|\uparrow\downarrow\rangle - q^{1/2}|\downarrow\uparrow\rangle)$

setting $\rho = |0\rangle\langle 0| = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -q^{-1} & 0 \\ 0 & -q & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and choosing for subsystem A the left spin gives

$\rho_A = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $S_A = \ln 2$ independently of q.

- In fact one should worry about left and right eigenstates $\left(\begin{cases} H|E_R\rangle = E|E_R\rangle \\ H^\dagger|E_L\rangle = E|E_L\rangle \end{cases}\right)$

with $|0_\mathrm{L}\rangle = \frac{1}{\sqrt{q+q^{-1}}}\left(q^{1/2}|\uparrow\downarrow\rangle - q^{-1/2}|\downarrow\uparrow\rangle\right)$

and define the density operator as a projector onto the ground state

$$\tilde{\rho} \equiv |0_\mathrm{R}\rangle\langle 0_\mathrm{L}| = \frac{1}{q+q^{-1}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & -1 & 0 \\ 0 & -1 & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Use now partial quantum (Jones) traces

$$\tilde{\rho}_A = \mathrm{Tr}_B\left(q^{-2\sigma_B^z}\tilde{\rho}\right) = \frac{1}{q+q^{-1}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and define a quantum group EE:

$$\tilde{S}_A = -\,\mathrm{Tr}\left(q^{2\sigma_A^z}\tilde{\rho}_A \ln\tilde{\rho}_A\right) = \ln(q+q^{-1})$$

This new quantity depends on q and is pleasantly related with the q-dimension of spin half rep.

Towards loop models.

- Recall the TL algebra

$$\begin{aligned} e_i^2 &= (q+q^{-1})e_i\,, \\ e_i e_{i\pm 1} e_i &= e_i\,, \\ [e_i, e_j] &= 0 \text{ for } |i-j| > 1 \end{aligned}$$

which admits a diagram representation with generators acting on lines and $e_i = \asymp$

closed loops having weight $n = q + q^{-1}$

- On two sites and for $H = -e_1$ the loop model ground state is $|0_\ell\rangle = \frac{1}{\sqrt{n}} \cup$

(normalized for the loop scalar product)

The loop density operator reads $\rho_\ell = \frac{1}{n} |0_\ell\rangle\langle 0_\ell| = \frac{1}{n} \asymp$

- The loop trace gives $\rho_{A,\ell} = \frac{1}{n} \, |$

and the loop entanglement $S_{A,\ell} = -\operatorname{Tr}(\rho_{A,\ell} \log \rho_{A,\ell}) = -n \times \frac{1}{n} \log \frac{1}{n} = \log n$

which is the same as the quantum group entanglement.

Claims.

- The QGEE can be defined more generally, and it obeys many interesting properties naturally required from entanglement. Its definition is particularly interesting when A is made of multiple segments (topological order and Hopf algebras?).

- The QGEE coincides with the loop model entanglement. We'll see soon what this means geometrically

CFT and QGEE

- The QG symmetric XXZ chain with $H = -\sum e_i$ is known to be critical with

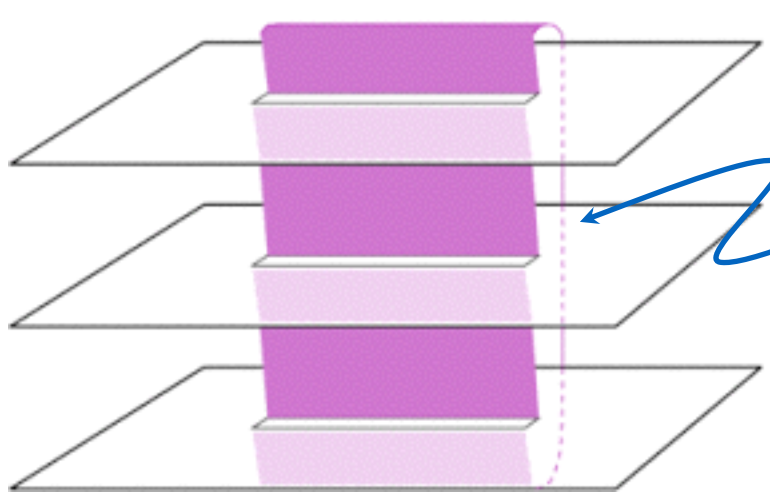
$$c = 1 - \frac{6}{x(x+1)} \qquad \text{parametrizing} \qquad q = \mathrm{e}^{i\pi/(x+1)}$$

- The ordinary entanglement is insensitive to the boundary terms and thus behaves like for ordinary XXZ with $c_{\mathrm{eff}} = 1$ (so in this case the expected result for non unitary models holds)

Let us study instesd the QGEE

- Replica trick: go to imaginary time and calculate the partition function on an N-sheeted Riemann surface [Cardy-Calabrese 2004-2016]

infinite sheets project onto L and R ground states so get $|0_{\mathrm{R}}\rangle\langle 0_{\mathrm{L}}|$



open lips build the partial density matrix (here A is interval of length L)

the model on the Riemann surface leads to

$$S^{(N)} \equiv \frac{1}{1-N} \ln \mathrm{Tr}\,\rho^N \qquad \text{the partition function}$$

To calculate the QGEE we use the loop formulation. The QG symmetric Hamiltonian gives rise, in the imaginary time version, to a model of (dense) self and mutually avoiding loops with fugacity n. In the plane, it is and old story that this model is described in the scaling limit by a free boson (Coulomb gas) with action

$$A[\phi] = \frac{g}{4\pi} \int \mathrm{d}^2 x \left[(\partial_x \phi)^2 + (\partial_y \phi)^2 \right]$$

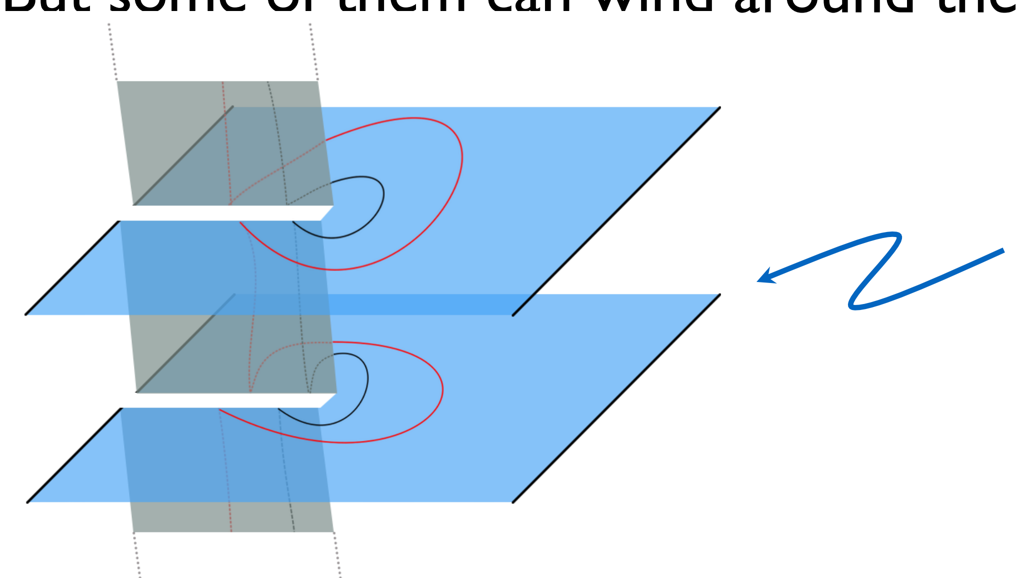
The boson arises from a SOS representation where loops are oriented (orientations summed over) and an arrow is interpreted as a wall between regions of different height.

In this picture, the microscopic weight n per loop is obtained by associating with every turn of an oriented loop a complex weight, eg $\mathrm{e}^{\pm i e_0/4}$ for the model on the square lattice. In the plane

$$\Delta N_{\pm} = \pm 4 \quad \text{gives} \quad n = 2\cos e_0$$

Finally, $g = 1 - e_0 = \frac{x}{x+1}$ and $c = 1 - 6\frac{e_0^2}{g}$

To calculate the Renyi entropy, we put the loop model on the Riemann surface. Now all loops must have the same weight n. But some of them can wind around the staircase(s)!



geometry with A on the edge

- The complex weights give the wrong fugacity $2\cos N\pi e_0$ to these non-contractible loops

- To calculate the scaling behavior of the partition function we must impose sewing conditions on N free bosons $\phi_j(z^+) = \phi_{j+1}(z^-)$

and correct the loop fugacity by inserting vertex operators at the extremities of the cut $\exp[ie_{l,r}(\phi_1 + \ldots + \phi_N)(z_{l,r}, \bar{z}_{l,r})]$

with $e_l = \frac{N-1}{N} e_0, \quad e_r = -\frac{N+1}{N} e_0$

Sewing conditions are implemented by forming linear combinations of the bosons, which see complex twists $\mathrm{e}^{2i\pi k/N}$

- The final result is unsurprisingly $Z^{(N)} \propto L^{-\frac{1}{6}\left(N-\frac{1}{N}\right)(1-6e_0^2/g)}$

and thus the Renyi entropy $\tilde{S}_L^{(N)} = \frac{N+1}{6N}\left[1 - \frac{6}{x(x+1)}\right] \ln L$

so QGEE scales with the real central charge

Entanglement in non-unitary minimal models

- The minimal diagonal CFT with $c = 1 - 6\frac{(p-p')^2}{pp'}$ can be obtained as the scaling limit of a RSOS lattice model with heights taking values $1, \ldots, p-1 \quad (p < p')$

- These models are "equivalent" to the loop model with $n = 2\cos\pi\frac{p-p'}{p}$ up to topological effects

- For the Renyi entropy in the ground state we can still use the Riemann surface approach (note: $|0_L\rangle = |0_R\rangle$ for these models)

  but now one needs to sum over sectors where non contractible loops get weights

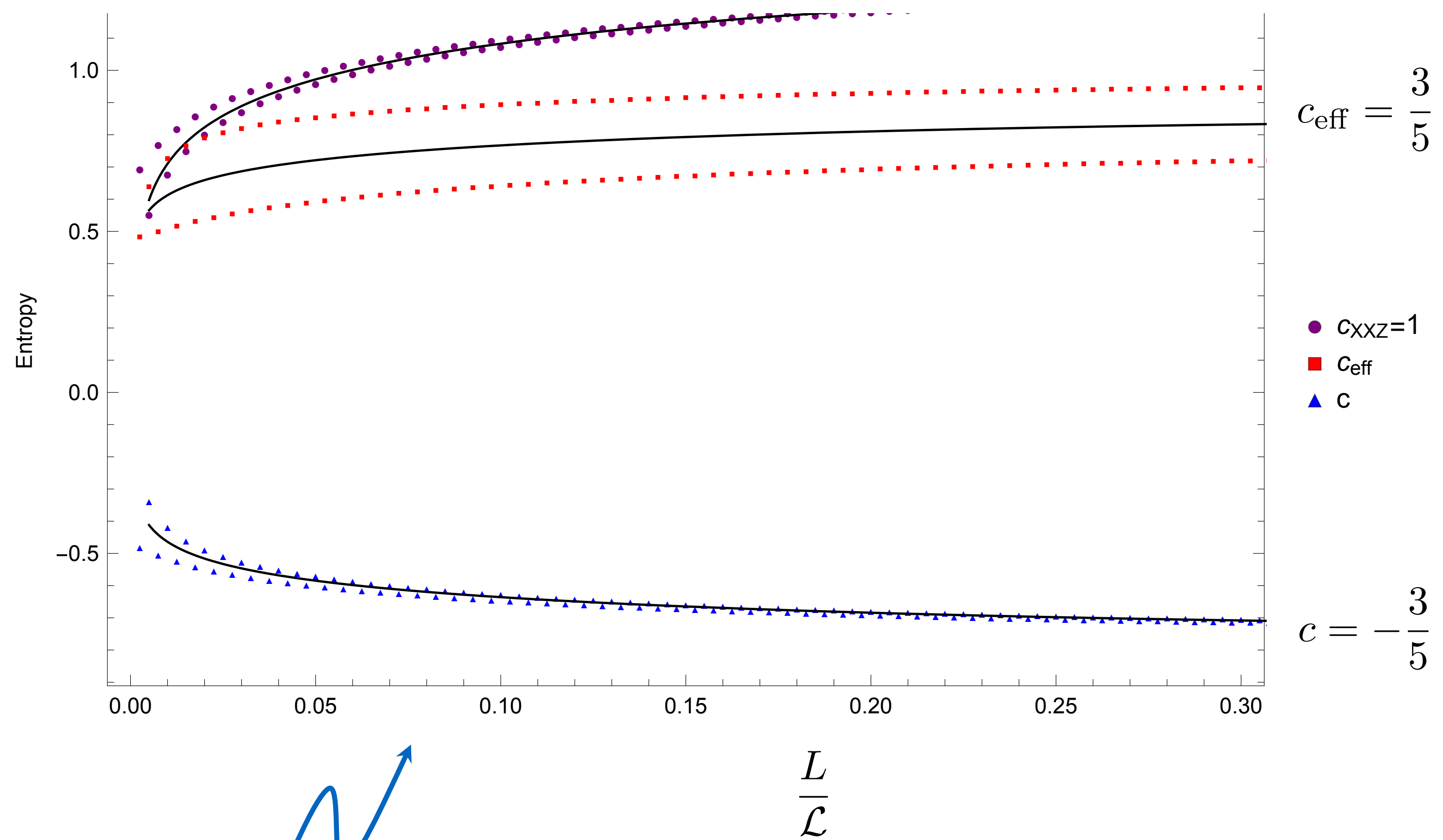
$$n_k = 2\cos\pi\frac{k}{p}, \quad k = 1, \ldots, p-1$$

  The sector with $k = 1$ dominates and gives c_{eff} as expected

  The sector with $k = p - p'$ (which would give c) is subleading.

- Many interesting questions: how to extract the real central charge for a given non-unitary theory using entanglement? What is the structure of the entanglement spectrum? Does one get the effective central charge for all boundary conditions?

Numerics:



fits to $S = \frac{c}{3} \ln \left[\frac{\mathcal{L}}{\pi} \sin \frac{\pi L}{\mathcal{L}} \right]$

Entanglement in super(algebra) symmetric models

- Motivation is to study susy chains whose central charge vanishes (partition function is trivial)

$$c = 0, \quad Z = 1$$

- Canonical example is the $sl(2/1)$ spin chain with alternating fundamental and conjugate (used to study percolation hulls of plateau transtion in class C of topological insulators [Gruzberg, Read 1999,Ludwig Read Saleur 2001]

In V states 1,2 are bosonic and state 3 is fermionic. Same in $\overline{V}$. Moreover, $\langle\bar{3}|\bar{3}\rangle = -1$

In subspace $|1\bar{1}\rangle, |2\bar{2}\rangle, |3\bar{3}\rangle$

$$e_1 = |0_\mathrm{R}\rangle\langle 0_\mathrm{L}| = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix} = \tilde{\rho}, \quad \mathrm{Str}\ \tilde{\rho} = 1$$

$$\tilde{\rho}_A = \mathrm{STr}_B\, \tilde{\rho} = |1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3|$$

$$\mathrm{STr}\, \tilde{\rho}_A^N = 1$$

If however we take traces instead of supertraces we get $\tilde{\rho}_A^N = \frac{1}{3^N} \left(|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| \right)$

and a non trivial result $\tilde{S}_A^{(N)} = \ln 3$

- Same definition applied to whole chain gives

$$c_{\text{eff}} = 1 + \frac{9}{\pi^2} \left(\log \frac{3+\sqrt{5}}{2} \right)^2 \sim 1.84464\ldots$$

Conclusions

There are questions specific to the non-unitary case. In particular about the behavior of the QGEE under RG flows

However elementary, the lattice approach opens the possibility to study a bunch of questions like:

- **Entanglement in chains with non-cocommutative Hopf algebra symmetries?**

- **Detailed structure of the entanglement spectrum in minimal CFTs/RSOS models (whose Hilbert space is not a tensor product)**

- **Entanglement when ground state is not normalizable? (eg, black hole sigma model)**

  the evidence is that it is related with the normalizable state of lowest energy

- **Entanglement when theory has a continuous spectrum?**

  the $m \to \infty$ limit of minimal models gives c=1 Liouville [Runkel Watts]

  and our approach gives

$$S = \frac{1}{3}\ln\frac{L}{a} + k\ln\ln\frac{L}{a}$$

??

- For loop models enthusiasts: EE in multicut situations (eg negativity) involves interesting topological problems…