# Spin chains for non compact (CFT) targets

# Compact and non compact targets

- In stat. mech. or solid state phys., spectra of critical exponents are usually discrete.

E.g. compactified free boson

$$H = \frac{1}{2}\int_0^L dx \left[(\partial_x \Phi)^2 + \Pi^2\right] \qquad \Phi \equiv \Phi + 2\pi r$$

$$h = \frac{1}{8\pi}\left(\frac{n}{r} + 2\pi r w\right)^2; \quad \bar{h} = \frac{1}{8\pi}\left(\frac{n}{r} - 2\pi r w\right)^2; \ n, w \text{ integers}$$

- Discreteness comes from the compact target. This can be seen well in the high temperature (mini superspace) approximation

x dependency neglected

$$H = \frac{1}{2L}\Pi_0^2 = -\frac{1}{2L}\frac{\partial^2}{(\partial\Phi_0)^2} \qquad [\Pi_0, \Phi_0] = i$$

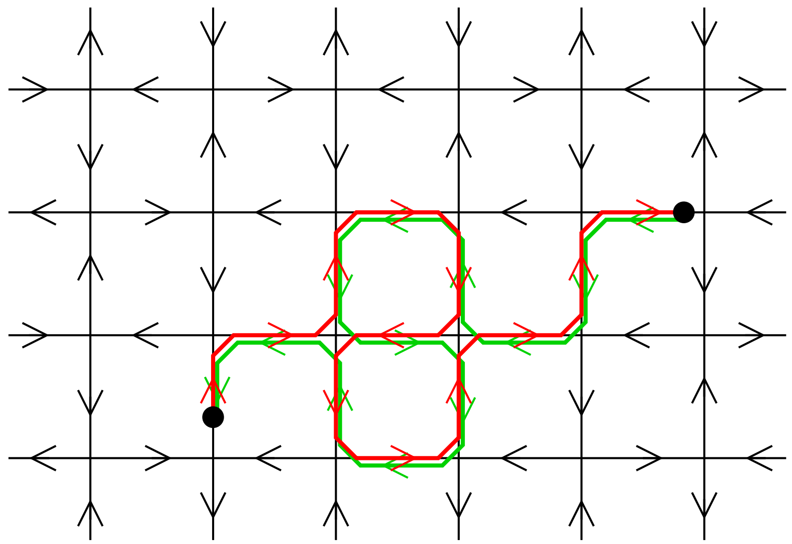
Hamiltonian becomes Laplacian on target $\quad \Psi(\Phi_0) \propto \exp\left(in\Phi_0/r\right) \qquad h + \bar{h} = \frac{n^2}{4\pi r^2}$

$$H = \frac{2\pi}{L}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right)$$

compact target implies discrete spectrum of critical exponents

- Several problems of stat. mech. or solid state phys. however involve non compact targets

For instance, IQHE plateau transition in the Chalker Coddington model. Calculation of $\overline{|G|^2}$ involves sum over paths with multiple overlaps



[Levine-Levy-Pruisken '83; Efetov '83;Chalker-Coddington '88; Weidenmueller '87; Read '89; Zirnbauer '99...]

Maps onto alternating $\mathrm{gl}(2|2)$ spin chain with infinite dimensional reps.

and onto the $\dfrac{\mathrm{U}(1,1|2)}{\mathrm{U}(1|1)\times\mathrm{U}(1|1)}$ sigma model at $\theta=\pi$

non compact target + supergroup symmetry!

- There's many other examples: spin quantum Hall, Nishimori point in disordered Ising, all geometrical problems with weak self-avoidance

- There's also many related examples on the AdS side of the AdS/CFT correspondence

- Some CFT results are available in the unitary case (Liouville, Witten's Euclidian black hole = sine Liouville). Little to nothing has been known up to now about the connection with lattice models; the RG flows; or the non unitary (supergroup) case

[Gawedzky; Dijkgraaf Verlinde Verlinde; Maldacena Ooguri; Teschner, Zamolodchikov^2; Fateev...]

Spontaneous Non compact symmetry breaking?

- Essential difficulty: lack of connection with stat. mech. techniques. In particular non compact integrable spin chains have resisted years of effort

[Faddeev Korchemsly; Belitsky et al.; Sklyanin;...]

- Main message: non compact (L,C)FTs can be studied using proper non Hermitian compact spin chains

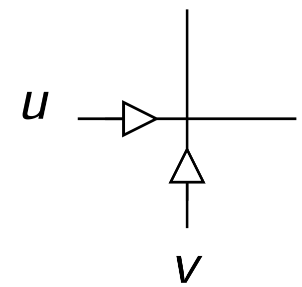
[Essler-Frahm-HS ’05; Jacobsen-HS ’06, Ikhlef-Jacobsen-HS ’08 ’12, Candu Ikhlef ’13]

# A lattice model for the black hole CFT

- Staggered 6 vertex model

  - $R$-matrix:

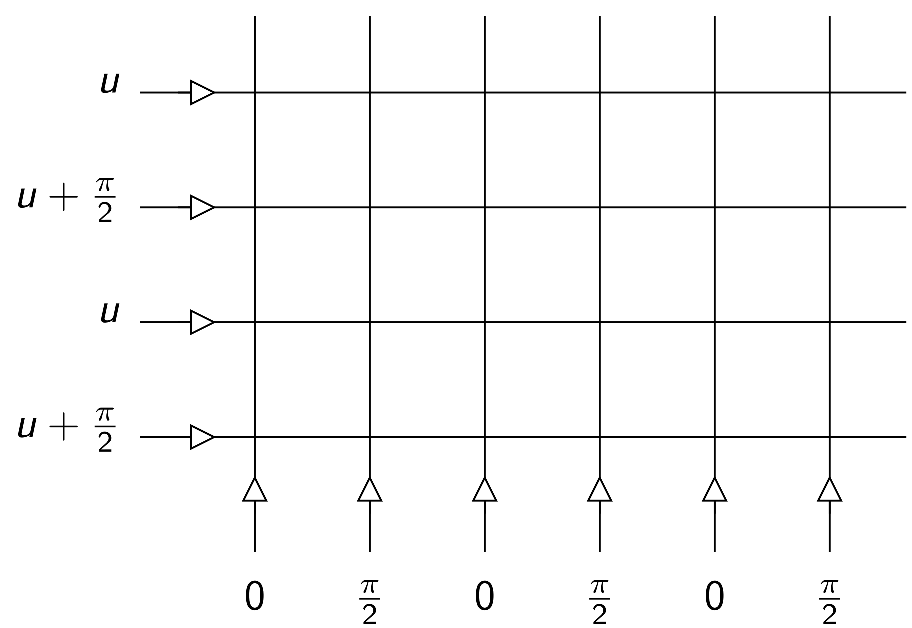
$$R(u) = \begin{pmatrix} a(u) & 0 & 0 & 0 \\ 0 & b(u) & e^{-iu}c & 0 \\ 0 & e^{+iu}c & b(u) & 0 \\ 0 & 0 & 0 & a(u) \end{pmatrix}, \qquad \begin{cases} a(u)=\sin(\gamma-u) \\ b(u)=\sin u \\ c=\sin\gamma \end{cases}$$

  - Rapidity lines: 

$:= \; R(u-v)$

  - Staggered spectral parameters

Baxter' solution of antiferromagnetic Potts model on square lattice



- Family of commuting transfer matrices

$$t(u) = \mathrm{Tr}_0 \left[ R_{0,2L}(u-\pi/2) R_{0,2L-1}(u) \dots R_{02}(u-\pi/2) R_{01}(u) \right]$$

Twice as many as for XXZ chain!

- Conserved quantities

$$Q^{(n)}_{\mathrm{ev}} = \left(\frac{d}{du}\right)^n \left[\log t(u) t\left(u+\frac{\pi}{2}\right)\right]\Big|_{u=0}$$
$$Q^{(n)}_{\mathrm{odd}} = \left(\frac{d}{du}\right)^n \left[\log t^{-1}(u) t\left(u+\frac{\pi}{2}\right)\right]\Big|_{u=0}$$

- Important examples
  - Hamiltonian $H = \frac{d}{du}\left[\log t(u) t\left(u+\frac{\pi}{2}\right)\right]\Big|_{u=0}$

$$H = \sum_{j=1}^{2L} \Big[ -\frac{1}{2} \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+2} + \sin^2\gamma\ \sigma^z_j \sigma^z_{j+1}$$
$$+ i \sin\gamma\ (\sigma^z_{j-1} - \sigma^z_{j+2})(\sigma^x_j \sigma^x_{j+1} + \sigma^y_j \sigma^y_{j+1}) \Big]$$

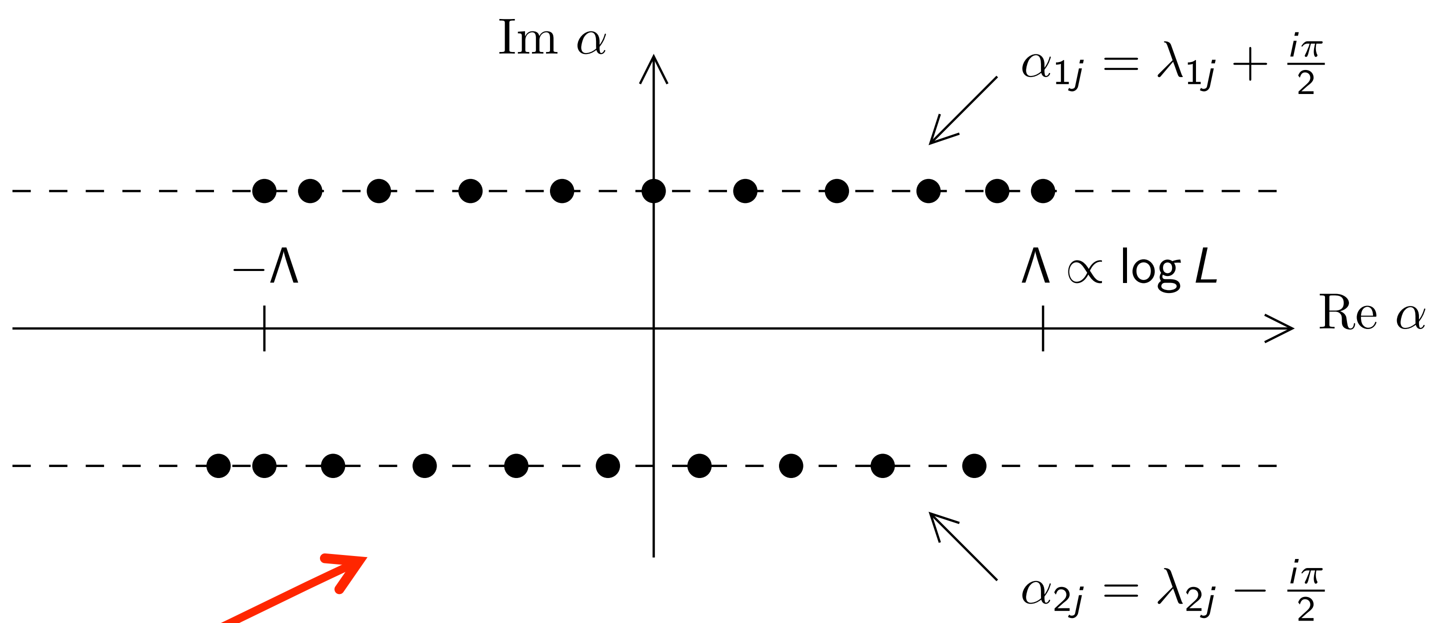
Finite, 2dim. reps.

  - Momentum $e^{-iP} \propto t(0) t(\pi/2)$
  - "Quasi-momentum" $S := \log\left[t^{-1}(0) t\left(\frac{\pi}{2}\right)\right]$

Ordinary XXZ would have factor one half

## Bethe ansatz solution:

- BA equations: $$\left[\frac{\sinh(\alpha_j + i\gamma)}{\sinh(\alpha_j - i\gamma)}\right]^L = -\prod_{\ell} \frac{\sinh(\frac{\alpha_j-\alpha_\ell}{2} + i\gamma)}{\sinh(\frac{\alpha_j-\alpha_\ell}{2} - i\gamma)}$$
- Energy: $$E = \sum_j \frac{2\sin^2 2\gamma}{\cosh 2\alpha_j - \cos 2\gamma}$$
- Low-energy states:




Two lines of roots in ground state: two holes quantum numbers $m_1, m_2$

- Study of finite size corrections $(\gamma = \pi/k,\ k \in ]2, +\infty[)$

Scaling of ground state energy: $c = 2$

Scaling of gaps (naive): $H = \frac{2\pi}{L}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right)$

$$\Delta + \bar{\Delta} = \frac{m^2}{2k} + \frac{ke^2}{2} + 0 \times \tilde{m}^2, \quad \begin{cases} e_1 = e_2 = e \\ m = m_1 + m_2 \\ \tilde{m} = m_1 - m_2 \end{cases}$$

$$\Delta + \bar{\Delta} = \infty \text{ if } e_1 \neq e_2$$

conformal ground state appears infinitely degenerate

$$p'(\lambda) = 2\pi\rho_a(\lambda) - \sum_b \int d\mu\ K_{ab}(\lambda - \mu)\rho_b(\mu)$$

$$\widehat{g}_{ab}(\omega) = 2\pi\delta_{ab} - \widehat{K}_{ab}(\omega)$$

$\widehat{g}_{\text{odd}}(0) = 0$

Eigenvalues of matrix $\widehat{g}(\omega)$:

$$\widehat{g}_{\text{ev}}(\omega) = \frac{2\sinh\gamma\omega\cosh(\pi/2-\gamma)\omega}{\sinh\pi\omega/2}, \quad \widehat{g}_{\text{odd}}(\omega) = \frac{2\sinh\gamma\omega\sinh(\pi/2-\gamma)\omega}{\cosh\pi\omega/2}$$

Scaling of gaps (less naive):

$$\Delta + \bar{\Delta} = \frac{m^2}{2k} + \frac{ke^2}{2} + 2\frac{s^2}{k-2}$$

continuous spectrum of critical exponents!

$$L \to \infty, \quad \tilde{m} \to \infty, \quad \tilde{m} = \frac{4s}{\pi}(\ln L + B)$$

$s \in \mathbb{R}$ is eigenvalue of $S := \log\left[t^{-1}(0)t\left(\frac{\pi}{2}\right)\right]$

- Identification of CFT. The model admits a natural massive deformation, with conserved quantities at all grades $Q^{(n)}_{\mathrm{odd}}, \ Q^{(n)}_{\mathrm{even}}$

This is reminiscent of complex sinh Gordon

$$A = \frac{k}{4\pi}\int d^2x\sqrt{h}h^{ij}\left(\frac{\partial_i z \partial_j^*}{1+|z|^2} + m^2|z|^2\right)$$

whose UV limit is related with Witten's Euclidian black hole CFT.

## 2D Euclidian black hole CFT

[Witten; Dijkgraaf Verlinde Verlinde; Bakas; Maldacena Ooguri; Troost et al....]

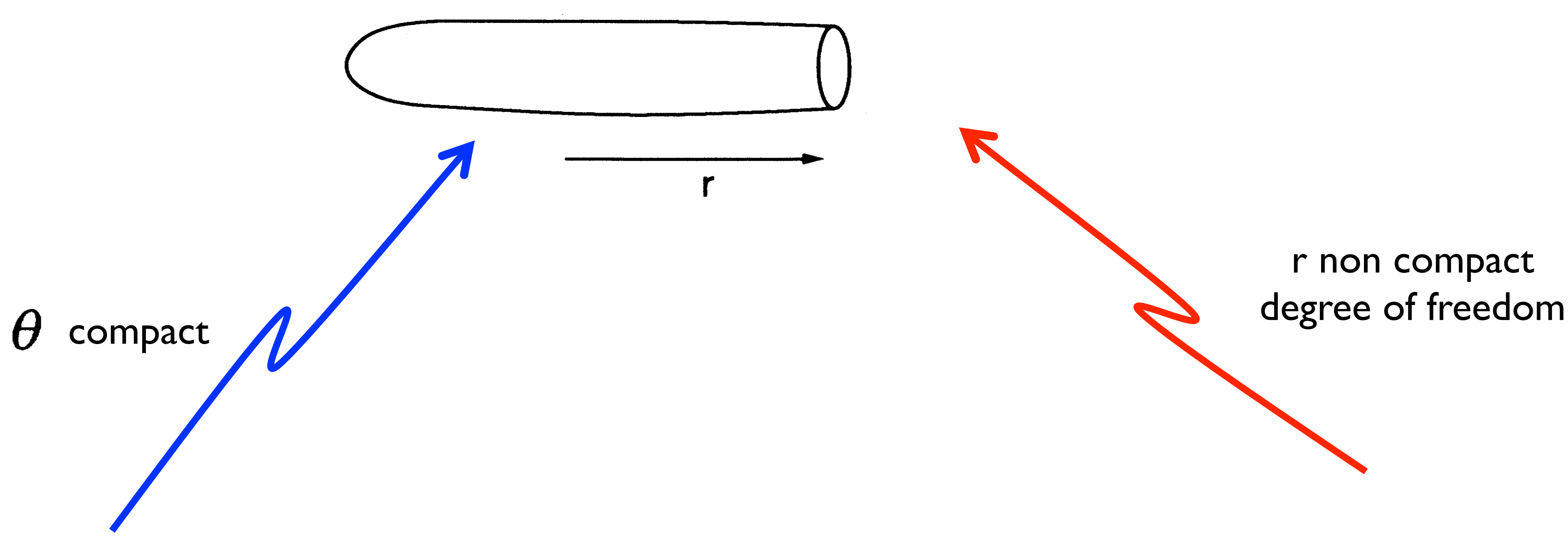
Gauged SL(2,$\mathbb{R}$)/U(1) WZW model

$$A = \frac{k}{4\pi}\int d^2x\sqrt{h}h^{ij}\left(\partial_i r\partial_j r + \tanh^2 r\partial_i\theta\partial_j\theta\right) - \frac{1}{8\pi}\int d^2x\sqrt{h}\Phi(r,\theta)R^{(2)} \qquad \Phi(r,\theta) = 2\ln\cosh r + \Phi_0$$

$R^{(2)}$ and $h$ worldsheet curvature and metric

Target space metric

$$ds^2 = \frac{k}{2}d\sigma^2,\ d\sigma^2 = (dr)^2 + \tanh^2 r(d\theta)^2$$



## Spectrum:

$$c = 2 + \frac{6}{k-2}$$

spin chain had c=2

$$h = -\frac{j(j+1)}{k-2} + \frac{(n+kw)^2}{4k}; \quad \bar{h} = -\frac{j(j+1)}{k-2} + \frac{(n-kw)^2}{4k}; \ n, w \text{ integers}$$

Normalizable states are mostly in the continuum (Laplacian on non compact target in mini superspace limit):

$$j = -\frac{1}{2} + is; \quad s \text{ real}$$

(s is momentum along the non compact direction)

$$\Rightarrow \quad h + \bar{h} = \frac{n^2}{2k} + \frac{kw^2}{2} + 2\frac{s^2}{k-2} + \frac{1}{4(k-2)}$$

almost the spin chain result but for this last term

$$(n, w) \leftrightarrow (e, m)$$

$$h = -\frac{j(j+1)}{k-2} + \frac{(n+kw)^2}{4k}; \quad \bar{h} = -\frac{j(j+1)}{k-2} + \frac{(n-kw)^2}{4k}; \ n, w \text{ integers}$$

$$\Rightarrow \quad h + \bar{h} = \frac{n^2}{2k} + \frac{kw^2}{2} + 2\frac{s^2}{k-2} + \frac{1}{4(k-2)}$$

almost the spin chain result but for this last term

the identity operator j=0 does not correspond to a normalizable state!

lowest normalizable state corresponds to s=0

$$c_{eff} = c - 24h_{min} = 2 + \frac{6}{k-2} - 24\frac{1}{4(k-2)} = 2$$

$$h - 24h_{min} = \Delta$$

- Of course to properly identify the continuum we need the density of states

## Known from CFT (Liouville reflection amplitude)

[Maldacena Ooguri; Teschner, Zamolodchikov^2; Fateev; Troost et al.;...]

$$\begin{aligned}\rho(s) &= \frac{2}{\pi}[-\log\epsilon + \partial_s(sB)]\\ B(s) &= \frac{1}{2s}\mathrm{Im}\,\log\left[\Gamma\left(\frac{1-m+ek}{2}-is\right)\Gamma\left(\frac{1-m-ek}{2}-is\right)\right]\end{aligned}$$

Liouville walk cutoff on the size of the target space

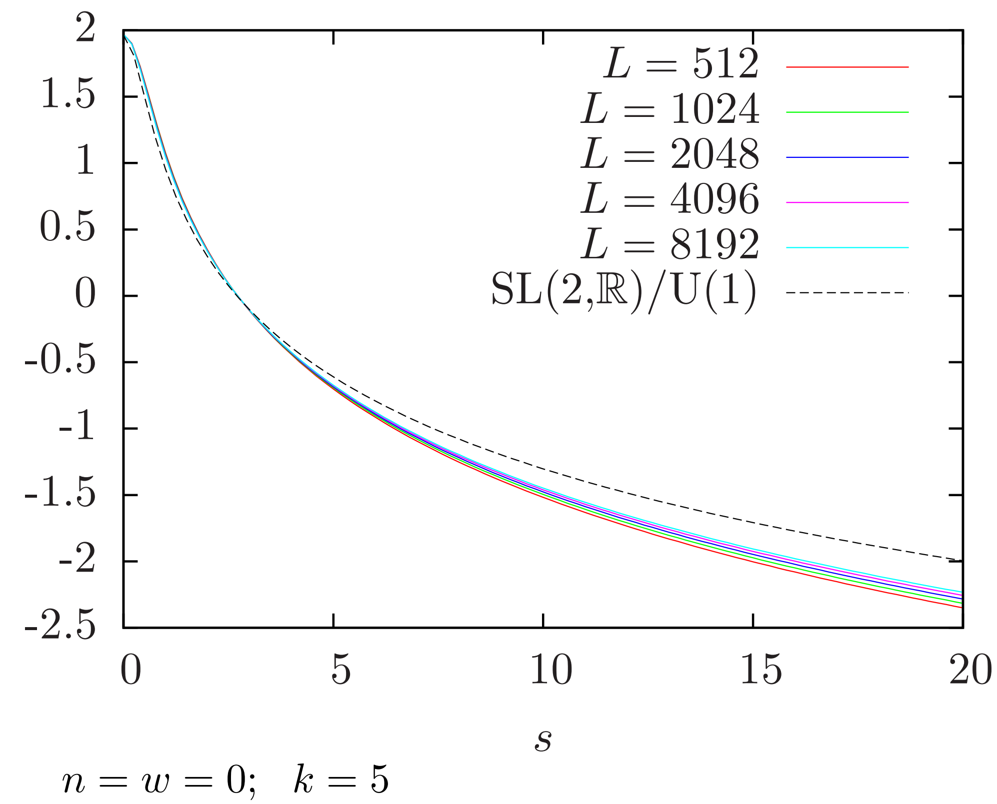
## Analytical work on the finite chain:

[Candu, Ikhlef, Jacobsen, HS]

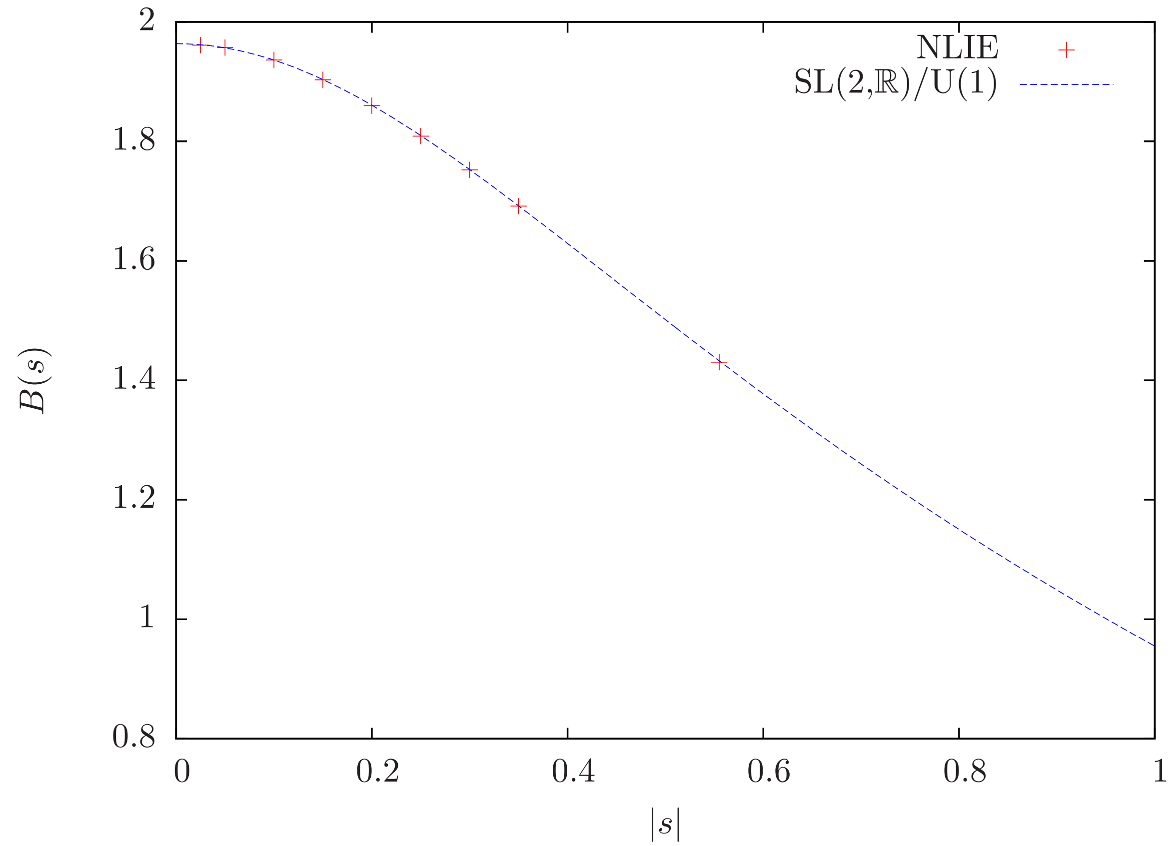
$$\rho(s) = \frac{2}{\pi}\left[\ln\frac{L}{L_0} + \partial_s(sB)\right]$$

## So far, B can only be determined numerically:

analysis of Bethe ansatz equations



analysis of non linear integral (DDV) equations




[Candu Ikhlef; Lukyanov HS]

$n = w = 0; \quad k = 5$

Normalizable discrete states: not visible in mini superspace limit

$$h = -\frac{j(j+1)}{k-2} + \frac{(n+kw)^2}{4k}; \quad \bar{h} = -\frac{j(j+1)}{k-2} + \frac{(n-kw)^2}{4k};\ n, w \text{ integers}$$

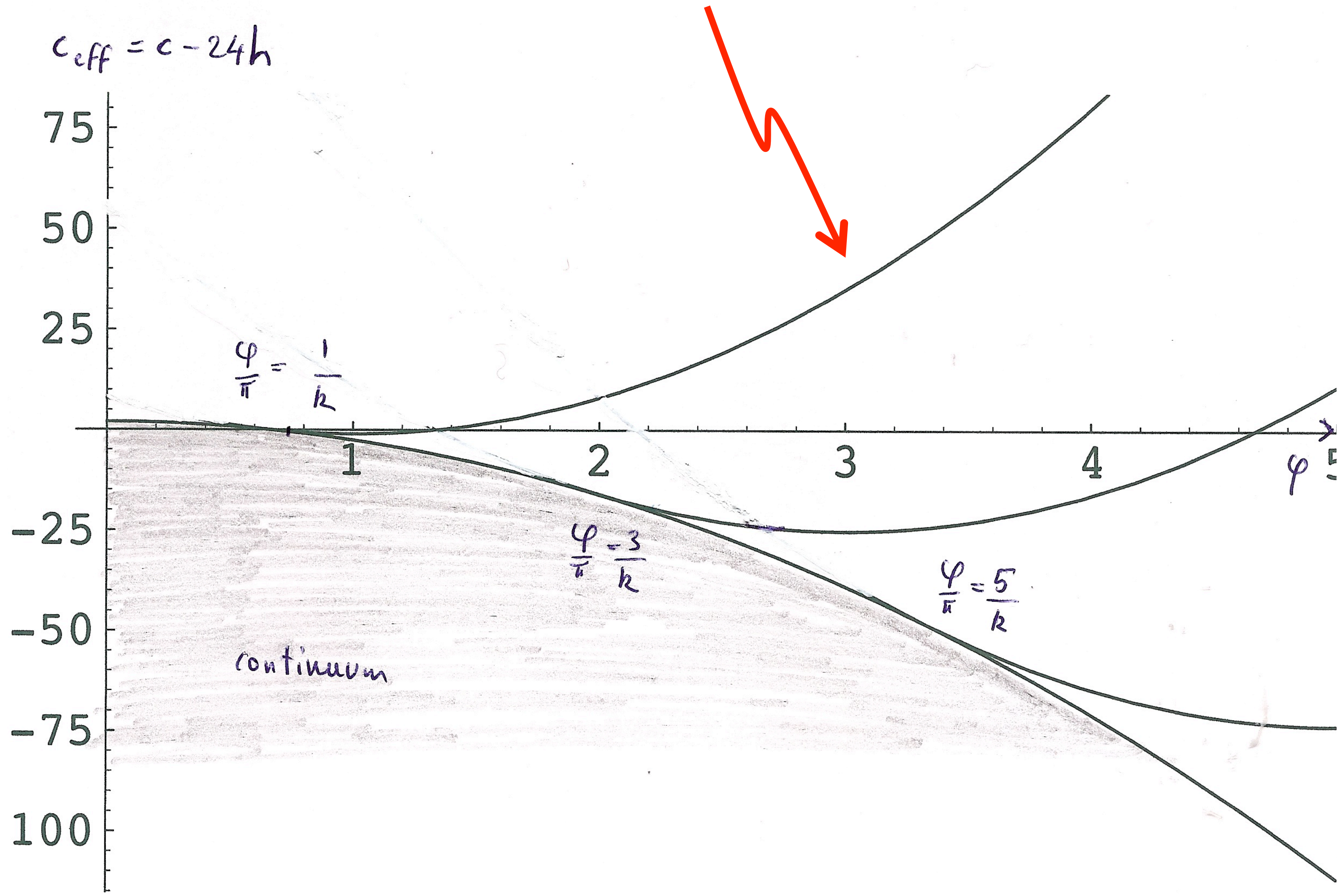
and j real variable $j \in \left[\frac{1-k}{2}, -\frac{1}{2}\right] \cap \mathbb{N} - \frac{1}{2}|kw| + \frac{1}{2}|n|$

It is possible to adjust w in the lattice model by turning on a twist in the Bethe equations (twisted b.c.)

$$w \equiv \frac{\varphi}{\pi}$$

while $n \equiv S^z$

As the twist increases, discrete levels pop out of the continuum effective ground state



# Conclusions

- Way to study massive deformations of non compact targets (complex sine-Gordon/ sinh-Gordon). Useful to understand issues such as non normalizability, discrete states etc
- We now know many more examples. They involve various cosets and supercosets (higher dimensional black holes, super black holes), and more or less generic microscopic models.
- The essential mechanism leading to non compact targets in the scaling limit remains mysterious
- Non compactness of the target appears also in the description of transport. Non hermitian QM is often used as a model for open quantum systems. How is this related?